MATHEMATICS TEACHING

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TEACHING AIDS IN
MATHEMATICS

★

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MATHEMATICS TEACHING

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NEW PUBLICATIONS

Applied Mathematics

By E. D. HODGE, B.SC., *and* B. G. J. WOOD, M.A., *Manchester Grammar School.*

This book provides a course in Mechanics covering the requirements of the various G.C.E. Examining Bodies, as set out in their Advanced Level Syllabuses in Mathematics and Physics. The authors assume that the reader is acquainted with elementary Algebra and Trigonometry and also with simple defferentiation and integration. Such differential equations as occur are, however, fully explained.

Chapters I-III deal purely with motion. Variable accelerations, including circular and simple harmonic motion, are introduced at this early stage in the hope that students will be convinced that all acceleration is not uniform—a very common misconception. Chapters IV-VIII deal with statics, and IX-XIII with dynamics of a particle.

In the remaining chapters, the authors consider rigid dynamics, which, though not required by all Examining Bodies in their Mathematics syllabuses, is a necessity for any physicist, and they also apply differential equations to physical and other problems resembling those of dynamics. Those who prefer to do so may start the book at Chapter IV with statics, and defer the reading of the first three chapters until ready to begin dynamics. It is also possible to leave the chapters on statics until after reading the dynamics of a particle, if that course is preferred.

With Answers **21s.**

School Trigonometry

By J. LARMOUR, M.A., *Sixth Form Master, Rugby School, and*
L. E. GODFREY-JONES, M.SC., *Head of the Mathematical Department, Rugby School*

This book is Part I of a comprehensive course in Trigonometry. It deals with basic principles and constitutes an easy introductory course suitable for beginners. The book will satisfy G.C.E. Ordinary Level requirements (elementary), and will incidentally be suitable for use in Preparatory Schools. Part II will cover remaining requirements for O and AO levels in both Elementary and Additional Mathematics, while Part III will complete the course up to G.C.E. Advanced level.

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Introductory Geometry

By C. WOIDE GODFREY, M.A.(CANTAB.); *and* R. C. B. TAIT, B.SC.(HONS.) (LONDON).

In this new edition a chapter has been added on Areas which deals with rectangles, parallelograms, triangles, circles, cylinders, etc. The book now supplies a two or three years' course for pupils who may not take Geometry in the Ordinary Level Certificate. They will, however, have sufficient knowledge of the fundamental facts to help them in their other courses. (Pupils following the Ordinary Level Syllabus should cover the ground in two years).

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EDITORIAL

THE ANALYSIS OF ERRORS

We learn by our mistakes; but it is better to learn from the mistakes of others. Nowhere is this more true than in the teaching profession, for if we study the mistakes which our pupils make we learn not only of their difficulties and shortcomings, but also very often of our own inadequacy in the teaching process.

For teaching is a two-way process involving an interchange of ideas between teacher and pupil and it is by their mutual understanding of each other's ideas that education takes place. Too often, however, this interchange does not take place, and instead the teacher tends to deliver instruction to the taught. This inevitably leads to an attempt on the part of the pupil to assimilate ideas which he does not fully understand; he tries to reproduce the ideas of his teacher in the form in which they are given to him rather than attempt to express his own views of a situation which he has comprehended. The result is the perpetration of a mistake due to limited understanding of the particular idea.

It is essential, therefore, when correcting mistakes to analyse these fully. More often than not a correction of an exercise simply draws the attention of the pupil to what has gone wrong with an implied caution of not to do it again. Too seldom does the detection of a mistake lead to an investigation of *why* it arose.

A recently published survey by L. G. Saad on *Understanding in Mathematics* (in collaboration with W. O. Storer; published by Oliver & Boyd; 15s.) surveys the answers given by over 3,000 children to questions put to them to test their understanding of fundamental ideas in mathematics. The author commences by pointing out that most classroom and examination tests are concerned primarily with attainment, which is not necessarily the same as understanding; in his survey he has been more concerned with understanding than with mere mechanical attainment. Thus he is more interested in finding what the child understands by a factor than in asking him to factorize a given expression in algebra.

It is possible to criticize some of Dr. Saad's questions as savouring too much of the usual run of 'grammar school' type questions and testing the children along familiar lines rather than finding their reaction to new situations. It is also possible to criticize some of his answers, for most of his questions have a selection of answers from which the child chooses the one which he feels is right. This, of course, is a weak point in the survey, for to understand requires one to express the ideas adequately in one's own words. However, granted that statistical necessity required the adoption of this method, the responses that Dr. Saad got, and his conclusions based on his analysis of them repay careful study. There are many examples of dependence on rule without any attempt to explain the reason for the rule. Thus questioned why the area of a 6 in. by 4 in. rectangle is 24 square inches, around fifty per cent of fourth and fifth year grammar school students tested gave as their reason "If we multiply the length by the breadth we get the area". And only about 30 per cent of the pupils could identify the graph of $y = 2x$ from four graphs shown. Again, over seventy per cent of pupils thought the statement "Alternate angles are equal" was always true, thereby giving a shocking revelation of some imprecise discussion in classrooms. In fact the poor results for geometry indicate that all is not well in the teaching of that subject.

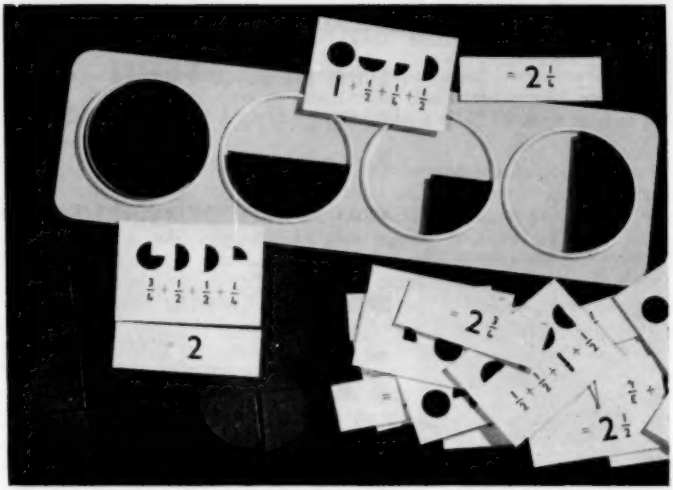
When it comes to the analysis of the results, Dr. Saad discerns certain psychological patterns of errors. Firstly he finds certain fixations: *verbal fixation*, where the child tends to state a rule rather than attempt an explanation; *passive fixation*, where the reason given for a mathematical process is the plain statement that it is the rule; and *action fixation*, where the manipulative part of a process has obscured the underlying reason for it. Next is noted the distracting effect of irrelevant material, the acceptance of a spatial picture instead of a mathematical concept, and meanings either being transposed, or being taken from special cases, or being amalgamated with facts.

When Dr. Saad analyses the effect of sex differences on the results he has obtained, it is surprising to find that in all of the three tests (in arithmetic, algebra and geometry) the understanding by the girls of the mathematical principles and concepts involved tends to lag behind that of the boys. It would be interesting to try to find reasons for this result; is it really a sex difference, pure and simple, or are other factors involved?

The author freely admits the limited nature of his survey and draws his conclusions with reservation, but reading this report one feels that there is a great need of a full investigation along the lines initiated by Dr. Saad. There is need to repeat the work on a larger scale, and to give the children greater freedom in stating their views. This should then lead to a fuller analysis both of the way the faults arise and of the methods needed to obviate them.

But more important, there is need for a more positive approach. Is it really sufficient to analyse errors? Rather should we not be making sure that we teach the right things at the right time and in the right way? Piaget has traced the growth of basic mathematical concepts in young children and has shown that many ideas develop much later than we believed. Might not the same be true of the secondary stage? Might we not be expecting the secondary school pupil to assimilate certain mathematical ideas at a stage when he is not ready for them psychologically? In fact the basis of our selection of what should appear in a secondary school course is largely tradition; the order in which we teach it is largely based on adult logic and not on child psychology. Examination syllabuses still play too large a part in determining the content of the secondary school course, yet what is contained in such a syllabus is not laid down by consideration of the child's development while at school; rather by the demands of entry standards to Universities and other external factors. Also, there is a clear need for investigation not only as far as present teaching of mathematics is concerned, but with an eye to the future. The content of the school mathematics course is clearly going to alter significantly in the next decade; are we certain we know what should be included and what discarded? As an instance, the past fifteen years has seen a swing away from formal geometry; is this a good thing, what—if anything—has replaced it, and what have we lost or gained in the process? Also, let us take the more successful parts of our teaching and seek the reasons for their success. For example, we in this country probably lead the world in the quality of our teaching of calculus; why should this be so?

When we begin to find the answers to questions such as these we shall perhaps be nearer to an understanding of why our teaching is sometimes ineffective, of why our pupils make mistakes, of how we can achieve better understanding in our pupils, and shall be on the way to overcoming the shortage of mathematicians and the present crisis in our subject.



Aspex Fractions Set 2

By P. B. Radnor

The object of this apparatus is to enable the young child to convert formal fraction sums into their concrete equivalents and then, by assembling the constituent pieces into 'wholes' and residual fractions, arrive at the answer by self-evident proof.

The set consists of a plastic inset board size 12½ x 3¾ ins. (with 4 recesses), a supply of loose fraction pieces and a series of cards (with the upper portions printed with fraction symbols and their corresponding cypher-forms underneath set out in sums) and answer strips to be matched to the sum cards. With the fraction pieces the child duplicates the sum on the cards. The pieces are then brought together to make 'wholes' with a fraction as the remainder. The child then finds the appropriate answer card and places it underneath the card bearing the sum. There are 12 cards in the set, each being plastic laminated on the printed face. Size of assembled pairs of cards 3¼ x 3 ins.

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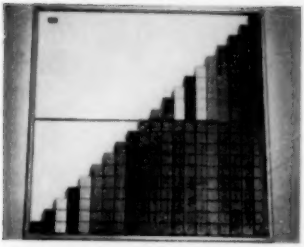

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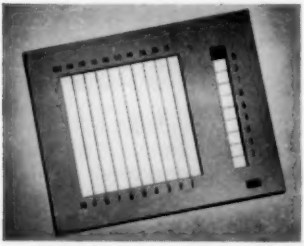

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SOME MECHANICS APPARATUS

FRANK MERRY

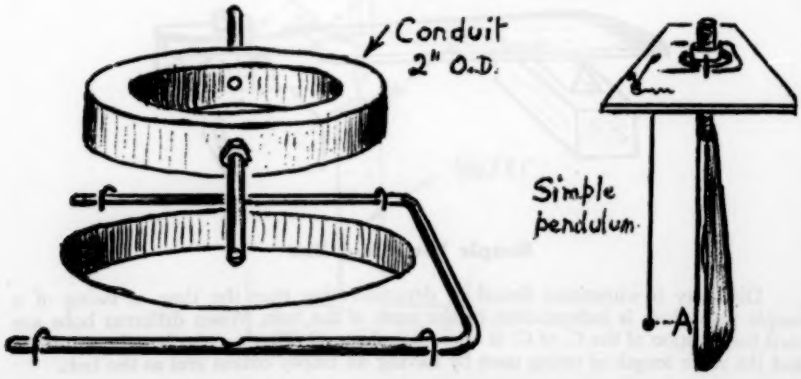
Is mechanics to be taught as physics or as applied mathematics? The writer (a teacher of physics) apologises in advance to those teachers of mathematics who dismiss mechanics as a branch of physics, but hopes the following models may prove useful to those who do teach mechanics.

When the movement in the teaching of mathematics is to present mathematical situations to the pupil in the form of 'visual aids', it seems appropriate that the student of mathematics should see his mechanics in action and not merely as diagrams and equations in a book. Pulleys, weights, spring balances and the like provide the material to illustrate the basic principles of the subject, but the apparatus described in this article is intended more to present the student with a problem to think about, to consider the application of known principles, and to discuss what their ultimate effect is likely to be. The best way to use such apparatus is to pose the question "*What is likely to happen if . . .* ". In some cases the student may be able to essay little more than a guess; in others he may be able to substantiate his ideas with a reasoned argument. In either case, there will have been some stimulus to thought.

Some of the apparatus is designed to 'stump' the student and the result may not be what he expects; he will have to think further to obtain a suitable explanation. Probably even the pure mathematicians will admit that the reasons for these results sometimes involve arithmetic, if not mathematics!

The ideas for many of the pieces of apparatus appear in an American book: *Demonstration experiments in Physics* by R. M. Sutton, published by McGraw-Hill Book Co., 1938.

Centre of Percussion of a Cricket Bat



A cricket bat or rounders bat is allowed to swing from a movable fulcrum made as shown. It works in V-shaped slots filed in thick wires fixed on opposite sides of a $2\frac{1}{2}$" diameter hole in an old drawing board.

A simple pendulum (cotton and bob) hangs through a smaller hole, countersunk on the under side.

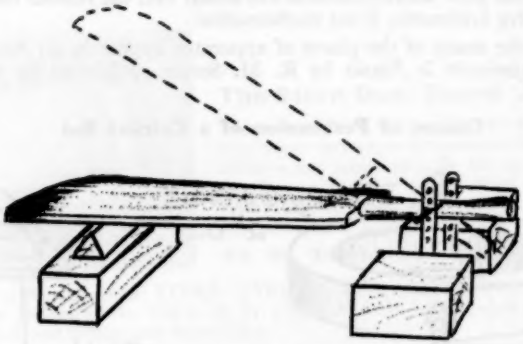
The bat is set swinging and the length of the simple pendulum is adjusted until it swings at the same rate as the bat (simple equivalent pendulum). This length is now marked on the face of the bat, at A.

A hammer blow at A makes the bat swing but the fulcrum remains in the slots. A blow above or below A makes the fulcrum shudder out of the slots.

The fulcrum is now moved to another position and the experiment repeated to find the new equivalent simple pendulum and hence the new centre of percussion.

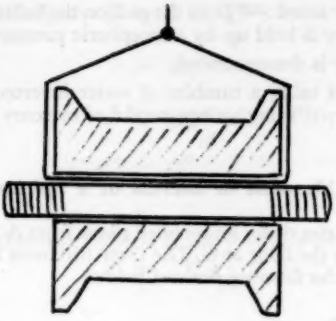
If the bat is an old one, holes may be drilled at corresponding centres of percussion and suspension and the reversibility of the compound pendulum demonstrated. (Drill the hole in the blade first.)

A further very striking demonstration with an old bat is to pivot it horizontally using a match-stalk as axle. If the bat is allowed to strike a triangular block of wood placed under the centre of percussion the stalk remains unbroken but snaps if the block is moved to another position.



Simple Pendulum Bob

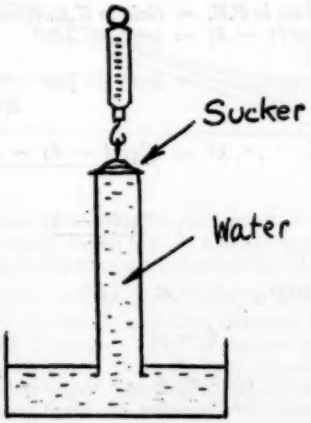
Difficulty is sometimes found in demonstrating that the time of swing of a simple pendulum is independent of the mass of the bob. When different bobs are used the position of the C. of G. is often also changed. This may be largely overcome and the same length of string used by having an empty cotton reel as the bob.



A piece of rod about 2″ long is sawn from a 6″ nail or other thicker material and this is screwed to take suitable nuts. By inserting this rod and then by adding nuts, several variations in mass may be made without altering the length of the pendulum.

Air Pressure

The apparatus shown in the diagram is set up. Any shape of bottle will do but a gas jar avoids complications!



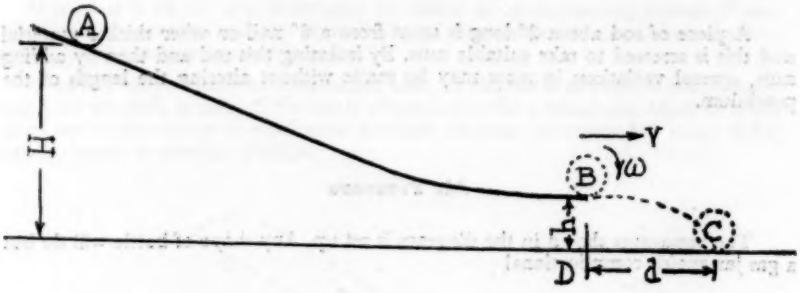
The question is now asked:—"Does the pull on the balance equal only the weight of the jar since the water is held up by atmospheric pressure?"

The correct answer is demonstrated.

The old experiment using a tumbler of water inverted over a piece of card is not easily explained. Try it using an egg-cupful of mercury!

Moment of Inertia of a Sphere

A ball is allowed to run down an inclined plane from A to B where it is projected horizontally and reaches the floor at C. The three distances H, h and d are measured and k^2 calculated from the formula derived below.



Theory:—

Loss in P.E. = *Gain in K.E.* (*linear and rotational*)

$$mg(H - h) = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}mv^2 + \tfrac{1}{2}mk^2 \frac{v^2}{R^2}$$

$$\therefore k^2 = \left\{\frac{2g(H - h) - v^2}{v^2}\right\} R^2$$

$$= \left\{\frac{2g(H - h)}{v^2} - 1\right\} R^2 \quad \ldots\ldots (A)$$

Resolving horizontally:— $(s = ut + \tfrac{1}{2}ft^2)$

$$d = vt$$

$$\therefore t = \frac{d}{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

Resolving vertically:— $(s = ut + \frac{1}{2}ft^2)$

$$h = \tfrac{1}{2}gt^2$$

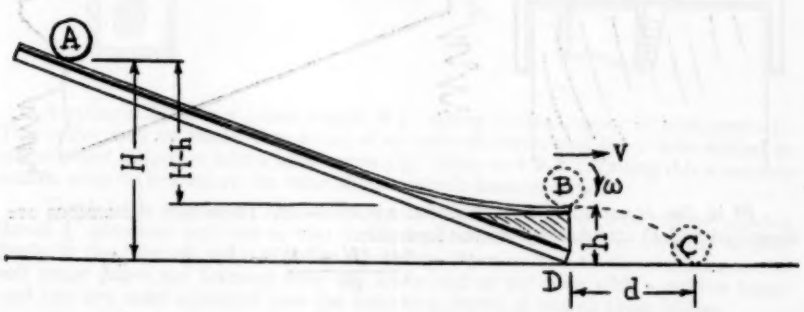
$$= \tfrac{1}{2}g\frac{d^2}{v^2} \text{ (from (B))}$$

$$\therefore v^2 = \frac{gd^2}{2h}$$

Substituting for v^2 in Equation (A) we have:—

$$k^2 = \left\{\frac{4h(H-h)}{d^2} - 1\right\} R^2$$

The apparatus to find k^2 consists of a simple inclined plane made by fixing a strip of $\frac{1}{8}$" hardboard 2" wide and about 4' long to a piece of wood 4' x 2" x 1". The lower end of the hardboard is, however, gently curved by means of a wedge of wood 2" wide about 8" long and slope 15° to 20°. This wedge is fixed to the lower end of the wooden plane and then the hardboard is fixed. Laths fixed to the edges of the plane prevent the balls from rolling off but only results from "free" runs should be observed.



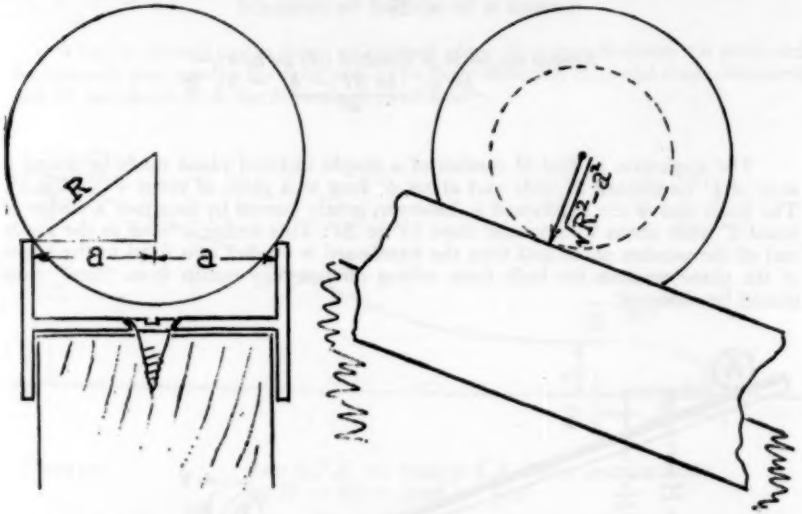
The lower end of the board rests on the bench or may, with advantage, be raised a few inches. The upper end is held in a retort stand and the height adjusted until the lower end of the inclined plane is horizontal. (Test with small spirit level).

A steel ball (diam. $\frac{3}{4}$" to 3") is placed at A, a point marked on the plane, a measured height H above the bench, and then released. It is projected horizontally and reaches the bench at C where a piece of carbon paper placed on a sheet of drawing paper pinned to the bench marks the spot. Several runs from the same point A are recorded on the same paper, the height BD (h) and the horizontal distance DC (d) are measured. From these the value of k^2 is found from the formula above.

Incidentally, the observed distance d should be approximately the same for *all spheres* released from the same point.

A refinement of the apparatus is the use of a track for the rolling sphere. The rotational energy of the sphere is also increased (at the expense of the linear energy).

The track is a 4′ long piece of curtain rail of true H-section (made by Harrison). This is screwed (using C.S. screws) to a thin board 4′ x 3″ x $\frac{1}{2}$″ (nominal), section as shown, with a wedge of similar wood as before.



R^2 in the equation is now replaced by $R^2 - a^2$. These two dimensions are measured and k^2 calculated from the formula

$$k^2 = \left\{\frac{4h\ (H - h)}{d^2} - 1\right\}\ (R^2 - a^2)$$

In this apparatus different spheres will give different values of d for the same values of H and h.

The first form of apparatus is useful for showing that all spheres have the same value for k^2 but the second form gives better results though these may have an apparent error of up to 10%. The experimental error is much less than this.

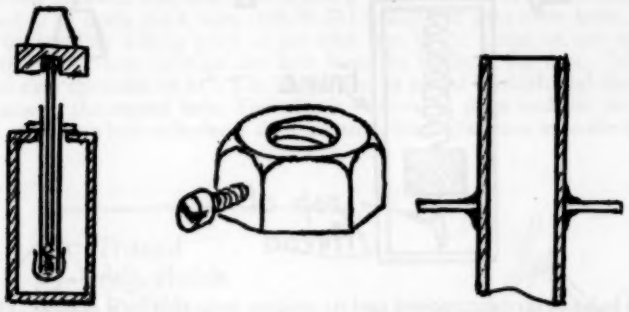
e.g. *Slope* 20°, $R = 1{\cdot}25''$, $a = 0{\cdot}237''$
$H = 17{\cdot}5''$, $h = 4{\cdot}4''$, $d = 12{\cdot}7''$

k^2 is found to be 0·65 compared with the theoretical answer 0·625 showing an apparent error of 4%.

The experimental term $\frac{4h\,(H - h)}{d^2}$ works out to 1·43 compared with the theoretical answer of 1·415 showing an experimental error of less than 2%.

Resolution of Forces

This experiment demonstrates the resolution of a vertical force into many oblique forces, each having a vertical and a horizontal component.



A collar is fixed to a 1 foot length of $\frac{1}{2}$" tubing (brass, copper or iron conduit). This collar may be made from a nut of suitable diameter and has a hole drilled in one side and tapped to take a small screw ($\frac{3}{16}$" Whit. or 4 B.A.). Failing this a suitable washer may be brazed on the tube about an inch from one end.

The tube is supported vertically from a strong stand. An iron rod or wooden dowel $\frac{3}{8}$" diameter and one or two inches longer than the tube is adjusted to slide freely in the tube. A piece of toffee wrapping paper or two or three thicknesses of soft tissue paper are fastened over the lower end of the tube with a rubber band, and fine dry sand is poured into the tube to a depth of two or three inches.

The iron rod is now introduced, and a wooden block with a $\frac{3}{8}$" hole bored part way through it is placed on top of the rod. Weights are applied to the block until the toffee paper breaks. If enough weights are not available (and a surprising weight is needed!) the wooden block is removed and the top of the iron rod struck with a hammer, gently at first and gradually increasing in force until the paper breaks.

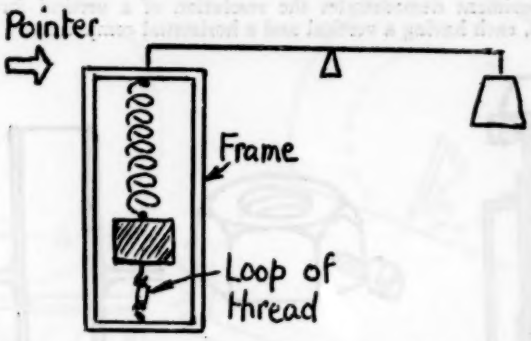
Most of the weight is supported by the vertical components of the forces between the tube and the particles of sand next to it.

It may be shown that without the sand the paper alone will support only a small weight.

For a quick demonstration the collar and stand may be dispensed with and the tube held by one hand near the top.

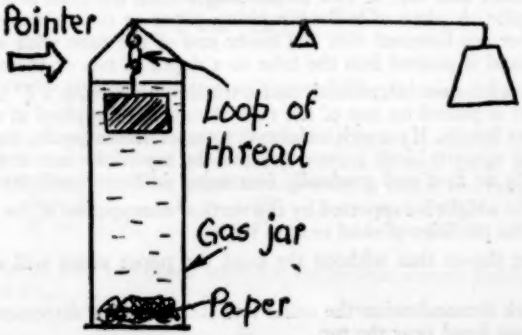
Acceleration on a Balance: Action and Reaction

(*A*) A wooden frame is placed upon one table of a platform balance or hung from a beam balance. Within the frame a heavy weight, about 4 lbs. or more, is suspended on a spring or elastic bands and pulled, stretching the spring farther, by a loop of thread hooked on to the bottom of the frame.



The balance is counterpoised and its position indicated by a pointer. The thread is burned and the movement of the balance observed.

(*B*) A weight or lump of lead about 500 gms. is suspended by a loop of thread from one arm of a beam balance which also carries a gas jar, or other tall cyclinder, containing enough water to cover the weight. The balance is counterpoised and its equilibrium position marked by a pointer as before.

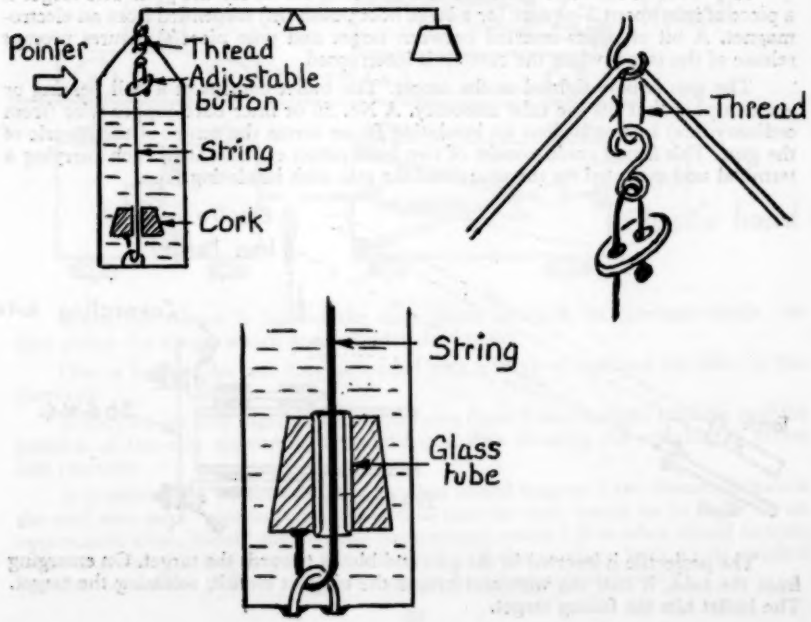


The thread is burned and there is an immediate rise in the arm that carried the weight but equilibrium is restored when the weight hits the bottom of the jar. Some screwed up paper at the bottom of the jar prevents any breakage.

After discussion of Demonstrations *A* and *B* the following question is asked:—"What would happen if a cork could be held at the bottom of the jar and then suddenly released so that it accelerated upwards?"

The next demonstration provides the answer and often surprises the student.

(*C*) A tall cylinder (Vim, Mirro, etc., scouring powder container) has the perforated top removed and two holes about $\frac{1}{2}$" apart drilled in the metal bottom. An inverted U of fairly thick wire (16S.W.G.) is soldered into these holes. This is managed by bending a long piece of the wire into the U shape at one end, and inserting the longer arm through one hole from the inside of the tube. (Make the U first and drill the holes to fit). The longer arm is pulled through and the shorter arm will now fit the second hole. The wire is soldered in place and the protruding pieces cut off. The whole cylinder is then treated with melted wax to make it waterproof.



A large cork (or two fastened together) is bored down the middle to be a tight fit on a $\frac{1}{4}$" diam. glass tube with smoothed ends. A piece of string is fixed to the bottom of the cork and threaded through the inverted U wire and through a button as shown, the final adjustment of the end knot being found later on.

A hook is closed round the loop and is held up by a loop of thread from one end of the beam balance. The correct length of the string is now found and the end knot tied, the button allowing for adjustments of the string when different cotton loops are used.

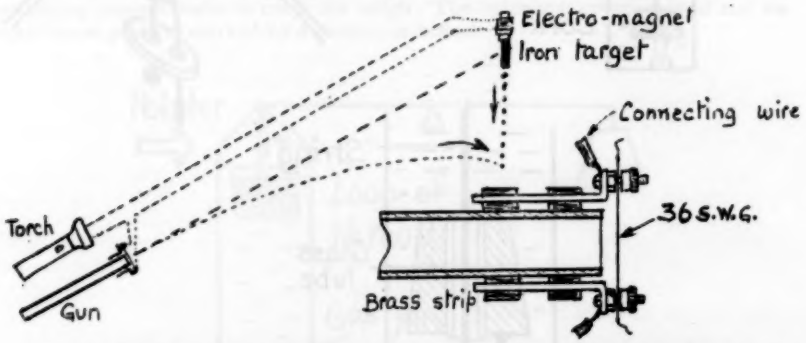
Water is poured into the cylinder and the cork pulled to the bottom by adjusting the button. The beam is counterpoised and the indicator placed in position. The thread is burned and the motion of the balance observed.

Most students think, at first, that this is another example of acceleration upwards as in Demonstration *A*, but as the *cork* accelerates upwards an equal volume but greater mass of *water* accelerates downwards. This is, therefore, another example of Demonstration *B*.

Falling Target

A $\frac{1}{2}$" tube of metal or glass about a foot long serves for the gun. The target is a piece of iron about 3" square (or a large boot polish tin) suspended from an electro-magnet. A bit of paper inserted between target and pole piece(s) insures prompt release of the target when the current is interrupted.

The gun is bore-sighted at the target. The bullet consists of a ball bearing or wooden plug that fits the tube smoothly. A No. 36 or finer bare copper wire (from ordinary flex) is stretched on an insulating frame across the centre of the muzzle of the gun. This frame could consist of two bent pieces of brass strip each carrying a terminal and mounted on the muzzle of the gun with insulating tape.

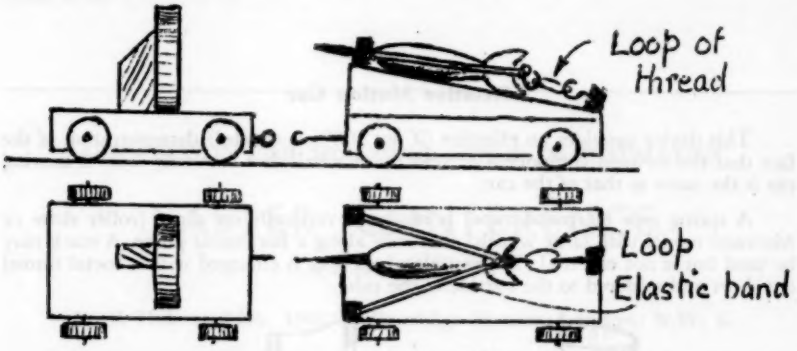


The projectile is inserted in the gun and blown towards the target. On emerging from the tube, it cuts the wire and breaks the magnet circuit, releasing the target. The bullet hits the falling target.

The experiment is equally successful with different angles of elevation of the gun and different muzzle velocities. One way to accomplish this is to mount the gun and magnet on a long spar of wood (clothes prop) so that, regardless of its angle of elevation, the gun is always aimed at the target. Terry clips are useful to hold the g un and a torch, the bulb of which is replaced by a broken bulb with wires soldered to the contacts. The whole thing is then self contained but care is needed to keep the hand supporting the spar out of the line of fire!

Action and Reaction: Conservation of Momentum

Two light cars are made from card, wood or tin cans. One carries a target about 4" diameter (made from rolled up wallpaper edging). The other has a sloping plane from which a dart *with plastic flights* is catapulted using elastic bands. These elastic bands carry a hook which is pulled and held taut by a loop of thread hooked onto another hook fixed to the back of the sloping plane.



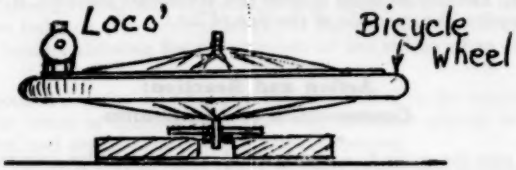
When the thread is burned the dart shoots forward, its carriage recoils, the dart strikes the target which moves under the impact.

One or both of the cars may be loaded with a lump of lead and the effect of this observed.

If the cars are tied together with stout wire there is only a slight shudder and the position of the cars appears to be unchanged thus showing the equality of action and reaction.

It is interesting, however, to discuss what would happen if the distance between the tied cars were considerably increased so that the dart would be in flight for an appreciable time. Would the carriage (as a whole) recoil? If so what would happen when the dart hit the target? Would this bring the carriage back to its original position? etc. (The C.G. of the whole system would remain stationary).

Conservation of angular momentum may be shown by mounting a circular "O" gauge track on a bicycle wheel free to revolve about a vertical axis. A toy locomotive (Hornby Clockwork works very well) runs on the track which moves in the opposite direction. By changing the loading on the loco' it is possible to control the velocities of both loco' and track relative to a fixed point.

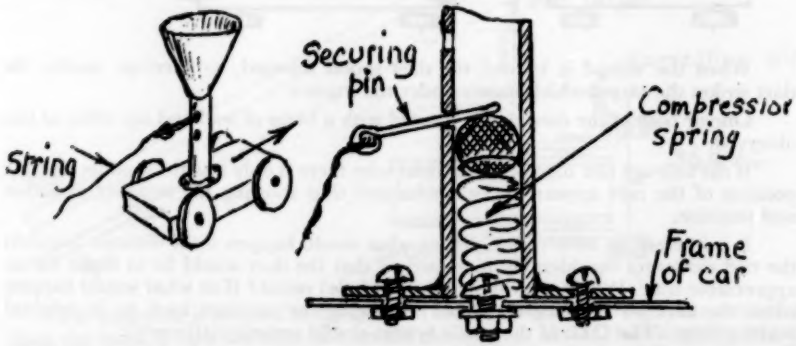


By cheating slightly (gently sloping the support holding the axle) the loco' may be made to remain stationary whilst the track revolves.

Relative Motion Car

This device provides an effective (if not 100% accurate!) demonstration of the fact that the forward component of velocity of a ball shot vertically from a moving car is the same as that of the car.

A spring gun ($\frac{1}{2}$" metal tube) is mounted vertically on a car (roller skate or Meccano model with large wheels) that rolls along a horizontal plane. A track may be used but is not essential. The muzzle of the gun is enlarged with a metal funnel cut short and soldered to the outside of the tube.



The spring, when compressed, is held in place by a pin which is pulled out when the car is pushed forward, shooting the bullet ($\frac{1}{4}$" diam, ball bearing) into the air.

The ball falls back into the car (quite often!) regardless of the velocity of the car.

By having two holes, one above the other, for the securing pin, two different heights of flight are possible.

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SUGGESTED SYLLABUS FOR PRIMARY SCHOOL MATHEMATICS

J. LLEWELLYN JONES

Spatial knowledge, involving experience in shape, form and size, which a child gains at the Infant stage should be extended throughout the Junior School course. In the early stages such activities would include the fitting together of pieces to make one whole shape, sorting and matching of shapes, etc., with the aim of encouraging perception and understanding. Simple craft has always offered the child opportunities of appreciating shapes whilst in his play and games there will be much to provide him with further experiences. From these simple beginnings it is hoped that the child's mind will eventually move towards an understanding of Mathematics in its more academic and abstract sense.

This supplementary syllabus of Primary School Mathematics was devised in an honest attempt to develop mathematical ideas and concepts by allowing children to investigate everyday problems in their immediate environment, using home made apparatus and material, thus helping to create a lasting interest in the subject. At the outset it is important to emphasize that it is not the intention to minimise the work done with regard to the basic principles of number, money, weight, etc. On the contrary the arithmetical side of mathematical teaching is of paramount importance; but we would be failing in our duty if children left us for the Secondary stage with endless computation as their only field of experience.

This scheme of work sets out to tackle the problem of introducing a much wider course of study and investigation to the child. One cannot claim that it provides all that might be required but any gifted teacher with enthusiasm for the subject will be able to add or subtract from the suggestions for assignments and centres of interest. The important thing is that there should be continuity and progression in all the work, for there is always the danger of dealing with unrelated topics in the guise of a syllabus of work. Furthermore we should guard against a scheme of work which confines itself to a narrow and unenterprising field offering little or no challenge to the enquiring mind of the average Junior School child.

We realise that the brighter the child the more readily will he understand the abstract ideas put forward by his teacher. This, however, does not mean that he should never be allowed to deal in the concrete. All pupils must be given the opportunity of purposeful activity and mathematical experience. The only difference between the treatment of bright and of less gifted children would be in the degree of difficulty in the assignments for practical work. There is no need to stress the fact that the latter, while taking part in this practical work, will carry out classroom tasks at a pace and level in keeping with their ability.

The normal child is always willing to accept the challenge of a well prepared assignment card and here much depends on the enterprise and skill of the teacher. Instruction cards may be given to each of six or more groups and each group will be given different problems to solve on one day, and on another they will be tackling the same problem in a slightly different way. They may disperse to various parts of the school playing area to carry out practical work with home-made apparatus and materials, record all their findings, work their calculations and finally return to the

classroom where the teacher will attempt to correlate the children's discoveries to bring about the full understanding of some mathematical experience. In this way children can be taught to integrate all experience, knowledge and language and to bring these to bear on new problems of future assignments. Only thus will the subject be made purposeful and fascinating to both teacher and child.

YEAR 1

Jig saw puzzles of mathematical shapes
Pattern making with definite shapes.
Geometrical templates of card and wood
Paper folding to obtain shapes
Comparison of lengths and widths—card of different lengths
Measurement of objects indoors and out-of-doors
Straight and curved lines
Work with simple Geo-Boards of nine pins and elastic bands
Bending wire to make simple shapes and later threading with cigarette shaped cylinders before bending. Thus four will make a square or diamond shape, six a hexagon, eight an octagon and so on.
Children encouraged to collect tins and boxes of different shapes.
The Square and the Rectangle
The drawing of simple plans, not strictly to scale to begin with
Measuring distance by means of (a) pacing (b) rope of given length (c) a ball of string (d) yardstick (e) using a home made wheel in a forked handle of one yard circumference (f) other means
Work in groups with Cuisenaire rods.

Year 1 *Projects* (a) THE HOME (b) THE POST OFFICE (c) THE BUS DRIVER.

YEAR 2

Lines—The horizontal and the perpendicular
Circles—Hoops, playground circles, radius, diameter and the centre, cylindrical and conical containers and the semi-circle
Geo-Board—Work with larger board of 25 pins
More difficult shapes—hexagon, octagon and pentagon
Use of Dowel rods to give the child ideas of solid shapes
Practical work in craft with wire and cylindrical paper rods
Triangles—Drawing different shaped triangles
Angles—the Right angle
—direction—making a simple Mariner's Compass Card
Plans and drawing to scale —1" = 1 foot, etc.
Work in groups with Cuisenaire rods.

Projects (a) HOLIDAYS—Where? How Far? By bus, train or car? How long? Costs, etc.
(b) SHOPS—Prices and costs of materials, weight and profit
(c) SCHOOL—Journey to and from, size, shape, registration, attendance, milk, meals, time table and playground
(d) THE LOCAL GARAGE—petrol, oil, tools and shapes, etc.

YEAR 3

Group work with Cuisenaire Rods
Further shapes—More complicated shapes on Geo-Boards, trapezium, parallelogram and irregular figures
The Circle—More difficult work—Introduction to the Compass
Solids—The Sphere, Cube, Prism, Cylinder and Cone.
Area—Geo-Board with 144 pins and 144 square inches. Thus finding the area of a variety of shapes both regular and irregular
Experience in area—The playground, school, classrooms and field
Recording of all measurements—Scale drawings
Further work with angles—Angles of the Set Squares, 45° and 60°
Angles of a triangle
Making a simple protractor
Introduction to simple compass bearings
Graphs—Simple line graphs based on recordings
Plans—The Plan of our school—Exercises in the correct reading of different plans
Map Work—Exercises based on the 1" Ordnance Survey Maps
The Map of Crawley

Projects (a) TRANSPORT—Walking, running, cycling, motoring, train, boat, and aeroplane
(b) WEATHER—Temperature, wind direction and velocity
Sunshine shadows and other recordings
(c) THE PARISH CHURCH—height of tower, plan of church, slope of roof, arches, area of site, shapes inside the church, church accounts and collections
(d) JOINERS (Stoners Ltd. near school)
What they do, the tools they use, e.g. lathe, mitre, T-square, gauges and rules
Measuring timber and constructing sheds

YEAR 4

More difficult graphs—(a) Costs of commodities against weight
(b) Speeds of trains, cars and aircraft, e.g. distance in miles, time in minutes
More about angles—Measuring angles—finding direction
Air and Sea routes
Volume—Practical work with different shaped containers
Problems based on knowledge of the rectangular solid, etc.
Triangles—Area covered—Similar triangles
Heights of buildings and other objects, with and without shadows
Finding distances by scale drawing and simple triangulation
Problems involving shape and area
Simple mathematical formulae, e.g. Circumference = 2 Pi × radius
Simple mathematics of space—Distances and speeds of planets
Circumferences of the planets
Relative speeds, etc.

The Mathematics of Motion—Comparative speeds—Objects falling, Projectiles and missiles—trajectory of a cricket ball—The wheel and axle—Simple pulleys—Gear ratios (Meccano models)—Pendulum and clocks—The motion of air, wind speeds—Motion of water—Flight

Suggested Topics for Further Study

The mathematics behind Farming—Building Houses
Television weather news—Time tables of all kinds
The Bicycle as a project—A visit to the Continent
The Air Mechanic at Gatwick—Engineer at Manor Royal, etc.

THE INTERNATIONAL COMMISSION AT KRAKOW

TREVOR FLETCHER

The 1960 meeting of the International Commission for the Study and Improvement of the Teaching of Mathematics took place at Krakow in Poland. The theme for study was *mathématiques de base*. It is not the function of the commission to reach formal conclusions and present agreed reports; it aims to raise questions in the minds of its members and to provoke thought on aspects of mathematics which they often do not have the opportunity to consider throughout the rest of the year. The members attend as individuals and not as delegates from their respective countries.

The phrase *mathématiques de base* can be understood in many ways; to some it means the logical foundations, to others the utilitarian topics which should be in the school course for every child, to others again it might mean the framework of elementary ideas which is the key to the pursuit of the higher levels of the subject at its present state of development. It was the aim of the conference to understand and to reconcile these many aspects.

The subject was introduced by Professor Papy, of the University of Brussels, Vice-President of the Commission. He explained how recent progress in mathematics had made necessary a reappraisal of its logical foundations. Traditionally mathematics developed by adding to the Greek body of knowledge, often giving to the pupil concepts which were weaker, and less general, than those which he could have assimilated. But modern mathematics demands a relaying of the foundations, and the *mathématiques de base* of the twentieth century is radically different from that of earlier times. The main outlines of the subject seen from this modern viewpoint are

analysis
algebra topology
set-theory.

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Professor Papy continued by showing how, in conformity with this scheme, elementary teaching of the subject could be based on the ideas of set theory, which provides a language for the subsequent development of many other branches of the subject. He showed how a particular kind of graph can be used to illustrate relations within a set, and how pupils can learn to appreciate logical structures and perform calculations with graphs of this kind. (Papy's graphs are a rich concept, related to Venn diagrams in logic and to the diagrams which Cayley used in group theory. Later the conference saw his theories in action in the classroom). We often confuse difficulties of verbal expression with difficulties of logical abstraction, and the use of graphical methods of reasoning enables these two difficulties to be separated.

After some sessions which were largely occupied by our Polish colleagues with a discussion of a new syllabus which was being proposed for their schools the main business of the conference proceeded in two ways—discussion groups and demonstration lessons. The main function of discussion groups is to help individuals to clarify their own thoughts, and there is little profit in detailed reports of their activities; but the demonstration lessons at Krakow raised many matters of general interest, and so most of the rest of this report is devoted to them.

It is easy to conceive a demonstration lesson as a lesson in which 'someone who knows' shows how it should be done. Many of the demonstrations which our own Association has arranged over the last few years have been of this kind, but a lesson can also be an experiment in front of an audience. A number of the lessons at Krakow were of this second type; not merely because of the incidental difficulties of language, but because the teacher was deliberately developing an unknown situation so that all who watched could learn from it, and learn from the mistakes and from the response of the class to unusual stimuli. This is a wider concept than a demonstration by an expert of something that he has fully worked out, and which he produces off pat; because the aim is to establish conditions from which new knowledge of the science of teaching mathematics can arise, and not merely to transmit certain pieces of mathematics by technical means which have been decided upon previously. We should make a special effort to promote this type of demonstration as well. They are, of course, very much harder to conduct than the more conventional kind.

Professor Papy's lesson was given to a class of sixteen girls, with the aid of an interpreter. His material was quite new. The pupils were asked to represent themselves on the board, which they did by drawing a set of crosses, one for each of them. Then they were asked to add crosses for their brothers and sisters, and then to add arrows denoting the property "has a sister". The problem was pitched at a level which they could understand, but thought was needed to solve it and to master the new notation which, in a great measure, they themselves devised. Some pupils' mistakes showed that relations could sometimes be deduced even when the people were not known, because the relations concerned had properties of (partial) symmetry and transitivity. When this was understood Papy drew a fresh diagram of an imaginary situation in which only some of the arrows were drawn, and the class had to decide what further arrows could be deduced.

This lesson was an obvious success. The topic provides a good exercise in logical reasoning, it gives an intuitive introduction to ideas which are fundamental in modern abstract algebra, it could be tackled, with obvious modifications, over a

wide range of ages, and it is not in the least dependent on traditional arithmetic. In Belgium, Papy is working out a full course on these lines, working with Ecole Normale pupils—girls of fifteen and over, often with reputedly low mathematical ability, who are training to be teachers. It is clear that they will be able to use much of the material which they are doing on theory of sets and topology with even younger pupils.

Dr. Gattegno gave a lesson in which he used the Nicolet film on the generation of conics as the loci of the centre of a circle passing through a fixed point and touching another circle. His questions were put with the help of an interpreter. As an experiment in communication it did not succeed. Dr. Gattegno has well known views on those who consider that the children are to blame if they do not learn when something is shown them; and this makes his lesson harder to appraise. The children were clearly on most unfamiliar ground. It was irrelevant that they 'had not done the geometry' leading up to the situation; the difficulty was that they just could not believe that they were wanted to describe the mathematics they had seen, in their own words, using their own judgement. They appeared to have been conditioned to an authoritarian type of teaching, knowing that a certain type of answer would please the teacher, and they were completely at a loss when they had to adopt their own standards of truth, stating what they had seen for themselves rather than repeating what they had been told. This lesson made no progress at all, in a forward direction, but some spectators may have seen that if further progress is to be made with many pupils who have followed the school course to the age of fifteen or sixteen the conditioning which they have undergone in the previous years must first be removed, as authoritarian teaching is inimical to true understanding. This lesson showed no one 'how to do it', but it *was* an experiment that led to a conclusion.

The first lesson by one of our Polish hosts was in great contrast. Mr. S. Turnau was working with pupils he knew, giving the next lesson in the course they were taking. There were no language problems, everything was as effortless and smoothly planned as one could wish, and it covered previously decided ground. (It was interesting that after the lesson, in private, Mr. Turnau expressed the fear that he had perhaps been wrong in giving a lesson that was a *demonstration* and not one that was an *experiment*). His class were given a geometrical pattern to study—photographs of a wallpaper design. It might have been a step nearer reality, and also more dramatic, to have given them an actual piece of wall paper, but this is a detail. The questions were directed towards finding the elements of symmetry of the design. How did these elements combine with one another? Which formed subgroups? Which families of symmetries did *not* form groups? Can you draw another unit cell with the same symmetry? Ten years ago it was most difficult to find an account of these ideas in English, yet this demonstration showed beyond any doubt that Sixth Formers can do group theory in a constructive and most worth-while way.

A lesson by Miss Felix, from France, author of a number of books on the teaching of modern mathematics was eagerly awaited; but it was, sad to say, a disappointment. An interpreter was in attendance, but Miss Felix had clearly underestimated the difficulties of communicating with a class of children who were not familiar with her style, and she had no plan in reserve when things broke down. This was a great pity, because her lesson on residue classes leading on to number rings is a good one, and it is a lesson that one would wish to see in a course of modern mathematics.

A large class of children from the earlier years of the secondary school assembled for a lesson from Madame Krygowska, who had made most of the local arrangements for the conference. Madame Krygowska is the leader of the movement for the modernisation of mathematical teaching in Poland, she has an imposing presence, and this lesson with eight or nine blackboards being used by the children resembled an oratorio performance under a virtuoso conductor. The subject was convexity, and the lesson was first rate. Ten children drew figures of their own—geometrical figures, doodles, faces and other recognisable objects. They were told to shade the interior of their figures. Then they were asked, "Take two points and draw the line joining them. Is this in the interior of your figure or not?" Clearly, different cases arose, and so the idea of convexity, and its precise definition, were born. The notion of the convex hull of a figure followed, constructed in many cases by placing a piece of elastic round plywood figures that had been stuck on to a baseboard. Once again we had a lesson that was clearly modern mathematics, which caused the class to think, and which would have found no place in the traditional syllabus.

I was late for the lesson given by Senor Galli of Uruguay, and for a while I did not realise that it had begun; for in a very few minutes he had managed to get the class thinking hard about a problem, and they were busy at it. He gave the lesson on the division of space into regions by different numbers of planes which is to be found in Vol. 1 of Polya's *Mathematics and Plausible Reasoning*. Knowing but a few words of Polish Senor Galli sought to make contact with his class by speaking a mixture of all the languages that he did know. His interpreter seemed taken aback by the technique, but the class managed easily enough. I have never known this lesson to fail, and this occasion was no exception. The class worked hard on ground that was new and where even the type of reasoning was unfamiliar. The lesson was marred only by the teacher's determination to finish in n dimensions or bust in the attempt! The final stages attempted too much too fast; but many teachers who watched will want to try the lesson for themselves.

Dr. Dienes's lesson was with young children of seven or eight. He used the teaching material of wooden blocks which he has devised and tested out in England, and which is to be the subject of a further extensive programme of work in the near future. The children were given a pile of blocks and set the task of forming other piles containing (in different cases) four, five or six times as much wood as they had been given. They then had to try to make an equivalent pile using as few pieces of wood as possible, given a large stock in the middle of the table to choose from. All this was aimed at cultivating an appreciation of different number bases and an understanding of 'carrying over'. Not being fixed to the base 10 the children were, effectively, learning to manipulate algebraic polynomials at the same time as they were mastering one of the early stages of arithmetic.

The six lessons illustrated various aspects of the central theme of the Vice-President's discourse. Many of the parts of mathematics which are considered modern and advanced can be understood by much younger pupils, they are basic in the part they play in the structure of mathematics as a logical discipline, and basic in the psychological development of the mathematician. These ideas must be better appreciated, and become part of everyday classroom practice.

Since this conference took place in Poland it inevitably raises the questions "What is it like behind the Iron Curtain?" and "How does their teaching differ

from ours?" I have ignored these matters in the earlier part of this report, because it is necessary to put them into perspective. Throughout the conference teaching mathematics came first, and it was only a secondary matter that we came from four countries of the East and five or six countries of the West. Our Polish hosts took immense pains to make our visit enjoyable and to show us something of their country. The University at Krakow provided excellent facilities, and we received every consideration.

Polish teachers are faced with an enormous task of reconstruction; there is the memory of the occupation when all but the most elementary teaching was prohibited, and when teachers ran clandestine schools under conditions of unbelievable difficulty. The country has new frontiers, they have few economic advantages to begin with, and they have the challenge of neighbours with highly developed technologies on both sides. All these circumstances make their problems different from our own; but if a criticism may be made in the friendliest possible spirit, it did seem that in spite of a revolution and a social philosophy which takes immense pride in being progressive the outlook of the teachers we met (with the most notable exception of Madame Krygowska and her pupils in Krakow) was very old fashioned. Furthermore we found nothing new in psychological knowledge or theory which referred to the teaching of the subject. In mitigation of this criticism it should be remembered that the permanent members of the International Commission are severe judges, having come together over the years because of their reforming zeal and because they find insufficient sympathy for their point of view in their own countries.

In making this first contact with the countries beyond the curtain the International Commission has made what could be a decisive step in its development. It is suggested that the meeting in summer 1961 should be in Austria, a country that is convenient for both East and West, and the topic proposed is "Le manuel scolaire". This title is intended to cover all aspects of the printed word in the presentation of mathematics, and not just the textbook. It offers much scope for preliminary investigation. How many of us discuss properly with our pupils books and their use? What techniques are involved in the writing of mathematics? What types of book are needed? In particular, what written material do we lack which members of the Commission could provide?

The British contingent to the meeting in Poland could have been far stronger, and it is a matter for regret that more of our countrymen do not play an active part in the life of the Commission. As the Commission's representative in Britain I am always pleased to meet any who would like to know more of its work or to take part in its activities; so why not visit Austria in 1961?

Enquiries about the 1961 meeting may be addressed to Mr. T. J. Fletcher, 148 Lennard Road, Beckenham, Kent, or to M. W. Servais, 60 rue des deportés, Morlanwelz, Belgium.

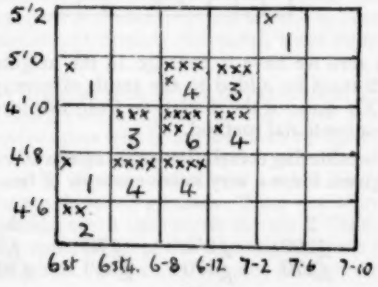
THE TEACHING OF STATISTICS — 'STAGE A'

CYRIL HOPE

Statistics means many things to different people. Generally it is a set or sets of numbers used to describe a situation in such a way as to ensure that some forecast can be made: it is diagnostic, prognostic and descriptive. The descriptive aspect tends to overlay the other two more important aspects. It generally includes some block graphs, histograms, pie diagrams and isotypes and as such is to be found masquerading as the more serious subject of which it is a very small part. To be prognostic or diagnostic, statistics demands a background of probability, which elevates it to a branch of respectable mathematics. If the subject is to be seen to have a future, probability must pervade all stages of its teaching.

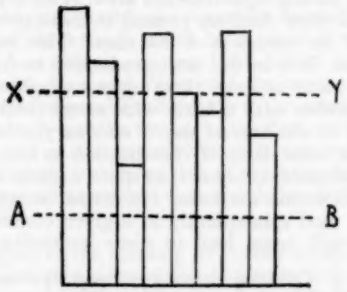
Like most other branches of mathematics, there has to be a 'stage A' in which the ideas grow almost unnoticed until there has developed such a body of experience crystallised into concepts and general principles, that a stage B is possible in which some sort of premises are postulated to enable a logical unfolding of the results to be carried out (even if this is only a 'loose' one). We shall be mainly concerned with these two stages in this article.

Junior children are encouraged to keep records of rainfall and temperature in graphical form. Occasional discussions will reveal trends but more important, they will raise questions of their reliability when used as a basis for forecasting. Will every 10th June have this same temperature? Is the recorded temperature typical of the day? etc. Many similar opportunities arise. A survey of the class will present 30 to 50 recorded ages. How shall we present them to give a quickly grasped conception of the ages of the members of the class? This leads to various graphical methods of presentation: it is in fact an introduction to frequency distributions. A few boys will have the same age and these suggest graphing frequency against age. If there are a large number with a fairly wide range (and you may always ensure this by taking the age to the nearest day if necessary), then the size of the graph paper will suggest that some form of classification to the nearest week, or to the nearest month would be more economical in space without losing the general picture of the *distribution* of ages within the form. The terms 'interval' and 'frequency' arise naturally and may be used subsequently in a given context. A scatter diagram of weight against height will again lead to some interesting discussions. Is there a



relationship between weight and height? You will have to choose your class carefully if you are to get stimulating graphs. Tabulation of heights in inches and weights in pounds will help if the worst arises. Playing about with this will enable you to get a boy to suggest that the averages of the two will help to come to a decision. There will be some talk as to whether if all boys were included it would be easier to arrive at a conclusion. How reliable is the sample? Comparison with the results of other forms would help to assess the reliability qualitatively. If the results of surveys are mounted and displayed in the room used for mathematics such comparisons between parallel forms is possible. Averages of body measurements raise the problem of standardised fittings for clothes. The frequency curves will suggest that certain measurements would fit a large number of children and scatter diagrams will suggest correlations between measurements so that a standard coat or pair of trousers for a boy could be suggested for a given age and height. All this would be carried out in a qualitative way without a great deal of calculation but relying on graphical interpretation for suggesting numbers and ranges which are credible. Of course girls could follow out this procedure too.

The calculation of an average lends itself to the development of the standard computer's method. If we graph each boy's height as a column in a column graph then the problem of average reduces to that of levelling out the skyscraper effect, using the peaks to fill the valleys. This emphasises the independence of the process of averaging and the zero level. One may take the averages of the 'deviation' from any datum line drawn in the figure.



Thus if we start from zero we have to average all the heights. If we start from AB, then the height of AB must be added to the result of averaging the heights above AB. If we start from XY some of the deviations from this level will be negative and so we approach the conventional method.

It is worth while considering correlation again against the knowledge of averages. Two sets of data are given. Here a very naive example of two commercial travellers:

week	1	2	3	4	5	6	7
A's takings:	£150	£200	£250	£50	£50	£100	£140
B's takings:	£170	£170	£170	£130	£100	£100	£100

If the two sets of data are compared graphically the difference in their sales is readily seen. A discussion as to which a company would prefer will increase critical faculties and prepare for the appreciation of the meaning and interpretation of distributions. It could well be that there was some correlation between their efforts, other causes accounting for the fluctuations in their takings. A scatter diagram would bring this out. It will have been noticed that each has the same average over the seven week period.

Other examples to try out frequency diagrams, averages and correlations are readily found: Each boy to estimate the weight and volume of a stone weighing about 20 pounds. The results should cluster round the true value and successive increases of interval length will show an optimum for bringing out the distribution of the weight estimates and its approximation to the Gaussian curve. A scatter diagram should bring out some correlation between the volume estimates and the weight. Calculation of averages would perhaps give rise to groans and demands to shorten the arithmetical labour. This is the stage at which the conventional method would be introduced and the development of stage B methods undertaken.

Any form of estimation will lead to valuable data for this type of analysis. Examination marks are another source of distribution, average and scattergram problems leading to plenty of discussion where each child has a chance of saying something original without necessarily being absurd. Some forms of sampling are treated essentially the same at this level. For example, each boy collects 20 samples of rye grass and counts the number of ears on each. This would lead to 600 results from 30 children. The counts can be graphed: frequency against number of ears. Usually two varieties are found showing as a two-humped frequency curve. Averages, symmetry of a frequency curve are all exhibited and used to find out the characteristics of the two varieties. Some petalled flowers have varieties distinguished by the number of petals and a similar experiment will produce similar problems. These are particularly illuminating in that the average is not necessarily the best value for the number of ears or petals. Some more discussion is called for about the normal error curve and the error in the mean and the reliability of the sample.

So far I have outlined one of the main strands of stage A which broadens out from something very simple until the stage at which important problems demanding more sophisticated techniques are conceivable and where problems are beginning to arise which simply cannot be satisfyingly resolved without more advanced methods. These demand a knowledge of probability if they are not to become so much mumbo-jumbo. Parallel with the strand already discussed, must come the development of ideas of probability.

Ideas of probability begin as soon as you have occasion to say "two out of three of you did . . . " What this means is a good starting point. Which three? and a nice discussion on how to deceive arises which may be taken up again as soon as percentages are introduced. One of the most important aspects of ratio, whether as 'fraction' or as 'percentage' is that of its use to symbolise a probability. The newspapers and Gallup polls are very happy hunting grounds: 'Nine out of every ten film stars use X'. Did they make a mistake when they chose the ten?; '50% of the electorate' but all the electorate did not vote; '95% of the washing machine manufacturers recommend . . . ' How many are there? Is it a true sample? Further discussion of the number

of blue eyed children in the school could be undertaken based on the sample provided by the class. A prediction could be made, one which could easily be checked by a count of the school. Absentees would give prominence to the insuperable problem of ensuring that all of a population are present at one time. All this is just simple introduction spread over a time and never carried to excess. The meaning of probability is understood in its empirical sense and the feeling is nurtured that, underlying the idea that if the sample is increased in size, so, more and more, will the probability of, for example, blue eyed children in the sample represent the proportion in the whole population. The terms 'population, sample, probability, sample space and sample point' may be introduced in a context which obviates definition.

More sophisticated ideas can be introduced now. 'I toss up a coin three times and it is heads once and tails twice. Is the coin loaded?' The possibilities may easily be tabulated: TTT, TTH, THT, HTT, THH, HTH, HHT, HHH. There are 8 possibilities. Of these three contain only one head. Hence the probability of one head only is 3 in 8 or 3/8. We have now to discuss the credibility or otherwise of the result in the face of the 'true' probability.

The pools give rise to other similar problems and are particularly fruitful in giving rise to the laws of probability. A match can be either 1, 2 or X. Hence the probability of a draw is 1/3. What is the probability that two matches will result in draws? It is easy to write out the nine possibilities and establish the probability of the event as 1/9. Three matches give 1/27 etc. It is easily inferred that the probability of n matches resulting in draws is $(1/3)^n$. The chance of getting the 'easy six' may be worked out, the 12 results and so on. The general law of multiplication of probabilities is easily formulated and its proof easily derived in class discussion.

Permutations and combinations follow from the same source. Crosswords from the popular Sunday papers give other problems on this multiplicative law.

Permutations may be introduced from the popular newspaper competition of putting the numbers 1, 2, 3 indicative of order of merit in some order opposite some attribute or qualification. The winner has to forecast the order the judges will choose. Suppose there are only three attributes. It is easy to see either by writing the possibilities in full or by arguing that there are 3 for the first, two for the second and the remaining one for the third that there are 3 x 2 x 1 possibilities. Gradually increasing the number leads inductively to the general formula of $n!$ possibilities. Letters of the alphabet can be used for drill examples. The question now arises of how many different ways you can choose two from three when ab is not the same as ba. An extension of the principle leads to the idea that if r are to be chosen from n, the first may be chosen in n ways the second in $(n - 1)$ and so on until for the last, having used up $(r - 1)$, we have only $n - (r - 1)$ left, and the familiar result follows.

$$^nP_r = n(n-1) \ldots\ldots (n-r+1).$$

What happens when the order ab does not matter? Start with a simple one such as 2 from 4. Call them a, b, c, d and write out the 12 pairs. If ab = ba etc. then it is readily seen that there are twice too many. $^4C_2 = 12/2$. Similarly $^4C_3 = 24/6$, and so on. An analysis of how this system works will suggest that there are $r!$ repetitions of each combination amongst the ^nP_r permutations. The general form can now follow to find immediate application in selecting eight teams from the pools and to estimating the chance of winning the treble chance assuming that there are on the

average 13 draws on the coupon of 52 matches. The result is of course

$$^{13}C_8 \;/\; ^{52}C_8$$

which reveals the futility of building one's immediate future on £250,000!

There is a tendency to build up a miscellaneous collection of examples related only by their dependence on a single principle. My preference is to avoid this by considering one problem field as a self contained topic developing the principles by particularly simple concrete examples. Throwing of dice, a pack of cards, may be used in this way. One of the topics will ultimately serve as a model for more complicated problems in other fields. It is therefore important to establish some sort of unity in its study so that its pattern may be discernible elsewhere.

So far we have woven two strands. The problem now is to consider ways of relating them; we want a connecting link between distribution, sample and probability, simple enough to permit of ad hoc treatment. The problem of tossing pennies is one such way in.

					Total
If we toss one penny we get		1 H	1 T		2
two	1 H^2	2 HT	1 T^2		2^2
three	1 H^3	3 H^2T	3 HT^2	1 T^3	2^3
	etc.				

The build up of Pascal's triangle follows and may be linked with combinations. If we toss up 2 pennies a number of times, then $\frac{1}{4}$ should be 2 heads, $\frac{1}{2}$ one head and one tail, $\frac{1}{4}$ two tails and so on.

For larger numbers of pennies the physical difficulties involved are almost insurmountable if one would check the validity of the assertions of Pascal's triangle by direct experiment. If we use drawing pins, calling pin up, 'heads', pin down 'tails,' then ten drawing pins to each of thirty boys enables 300 results to be obtained very quickly. The results may be tabulated:

	H^{10}	H^9T	H^8T^2	etc.	Total
No. of occurrences	1	3	15	etc.	300
Experimental probability	1/300	1/100	1/20	etc.	

These may be compared with the theoretical probabilities. A histogram using the number of heads on the base scale and the experimental probability on the frequency scale, will show the familiar Gaussian or normal probability curve. If curves are drawn for two drawing pins, three, four, ... 15 . . . etc. then the gradual approach to continuity is seen intuitively and, at this stage, left at that.

A more tractable experiment is to use packs of cards from which are dealt hands of two cards, three cards, etc. in each case counting the number of reds in each hand. This is analogous to the coin problem and tabulated results together with frequency probability diagrams will help to prepare the way for the normal probability curve.

We have now more or less covered all the elementary ideas underlying a more detailed study of statistics. The above work does not necessarily represent a continuous programme but will be spread over two or three years. Some of it may be

done by way of recreation, useful in motivating practice in elementary arithmetic. At no time should calculation be so involved that its boredom and effort obscures the lightness, elegance and simplicity of many of the principles involved.

The stage is set for a proper study of this branch of mathematics. More complex and practical examples from commerce, biology, physics and sociology are demanded. A later article will consider these examples to show how statistics may be developed without too much computation.

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TEACHING AIDS AND LOGIC

V — PUNCHED CARDS, AND CONCLUSIONS

TREVOR FLETCHER

The last three articles of this series have been concerned with electrical models which illustrate the laws of Boole Algebra, and which may be designed by using it. We now describe a 'model' consisting of a set of punched cards which provides another excellent illustration of the algebra, and which can be used to verify its laws and to solve logical puzzles of a type that is popular in newspapers and magazines. The one set of cards can be used to solve many different puzzles.

Business houses not only use complicated electrical machinery for stock control, payroll and similar operations, for many years they have also used punched cards. The little electrical machines described earlier in this series are the classroom versions of the business houses' electrical computers, and we now describe the classroom punched card system.

We use a pack of cards with punched holes round the edge. These holes can be cut away as desired to give slots instead of holes. If a needle is inserted through the holes in one particular position and then lifted, the cards with a hole in this position are raised up and the cards with a slot are left behind. If we denote the cards with a hole in the position by a, then the set a is removed by the needle and the set a' is left behind. It is immediately obvious how one may, by working successively on two positions on the card, extract the set $a + b$ or the set ab. This pack of cards can be used to verify all the laws of Boole algebra. Suitable cards can be purchased from suppliers of office equipment or with a little trouble be manufactured oneself. However it is not easy to make a home-made pack with the same accuracy as the commercial article, and it is very desirable that the cards should run freely over one another and not stick because of slight irregularities in the punched holes. Commercial cards carry a hundred or more holes round the edges, but when punching a pack for use in set theory one usually does not need more than, say, two dozen positions at the most.

Different systems of punching can be used for different purposes, but the most important system is the one with 2^n cards punched with all the possible arrangements of hole and slot on n holes. If we regard the slots as signifying 0 and the holes as signifying 1 then the 2^n cards are coded with the first 2^n numbers in the binary system. For an educational model it is usually sufficient, at least at the beginning, to work with $n = 4$. The cards then begin 0000, 0001, 0010, 0011, etc. up to 1111. Later this set of 16 cards can be increased to 32 or 64 for more complicated problems.

For some purposes it is a good plan to punch the holes on the bottom of the card with the complementary coding, i.e. underneath 0110 punch 1001. This brings out de Morgan's laws in an interesting way, as it can be seen that an *and* operation on the holes at the top of the cards is inevitably accompanied by an *or* operation on the holes at the bottom.

This pack of cards is a rudimentary 'logical abacus', and if the time comes when set theory is taught in the primary school this pack of cards could well play the part which number rods now play in teaching the elements of arithmetic.

Tautologies of the algebra can be verified, as they are expressions which remove all the cards from their box. These cards have the structure described in abstract algebra as a distributive lattice, and the simple system we have just described can be extended in many ways. Projective geometry can be regarded as a branch of lattice theory, and a model of a finite geometry can be constructed by using cards to represent the lines of the geometry and holes to represent the points. I have, for example, a set of 31 cards punched as the 31 point projective geometry. Every theorem in the geometry can be interpreted as a theorem about sorting the cards; but it is not altogether easy to extract, say, six cards which envelop a conic.

This is only the beginning. We have merely described the two simplest sorting operations on edge-punched cards. It is not difficult to devise other sorting operations which, for example, do not obey the distributive law and so to produce models of a wide range of other abstract systems of the type which are studied in modern algebra. I know of no full discussion of the potentialities of punched cards used in this way.

The more practically inclined student will wish to put the cards to other uses, for they can be used to solve a wide variety of logical puzzles; and with practice they may be quicker than pencil and paper.

A solemn treatise on the different methods which one might employ would be out of place, but we give below a short selection of puzzles which can be solved by manipulations with a pack of cards and suggest the line of attack in each case.

EXAMPLE 1. *Alec, Bill and Cecil are witnesses to a robbery; but unfortunately their evidence is not very reliable. The suspects are Jones, Robinson and Smith. Alec says, "Smith and Robinson did it, Jones is innocent." Bill says, "Smith and Jones did it, Robinson is innocent." Cecil says, "Smith, Jones and Robinson are all guilty." Now I happen to know that Alec and Bill are each correct about only one suspect, whilst Cecil is definitely a better witness than either Alec or Bill.*

Who committed the robbery?

This is a simple example, a set of 2^3 cards is enough. Use the three places to denote the three propositions 'Jones is guilty', 'Robinson is guilty' and 'Smith is guilty' respectively.

EXAMPLE 2. *I am planning to take some of my nephews and nieces out for a picnic. There are five of them, but I can only manage to take three. Now:—*

Connie won't go with Donald or Alice,
Eric will only go if Alice is there,
Brenda won't go with Donald unless Eric is there too,
Alice won't go if the boys are together.
What must I do?

This can be worked using five positions on 2^5 cards. After initial "and-or" sorting to remove the cards corresponding to combinations of children which are not wanted, a final manual search through those left in the pack is needed to extract the card (or cards) with just three holes. Commercial cards have far more holes than we normally need for this type of problem, and there are many ways by which the basic system of punching can be extended. For example, for puzzles like this it

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would be useful to use further positions on the card to denote the number of holes (as opposed to slots) appearing in the basic positions. This would avoid the final operation of searching for cards with three holes. But these refinements are not essential, and anyone who experiments with the basic system will soon devise further tricks of punching for himself.

The next puzzle may be solved in a similar way.

EXAMPLE 3. *Jack, Joe, George, Jim and Gerry are journalists or judges. Now journalists always tell the truth, but judges always lie.*

Jim says Gerry is a journalist.
Joe says Jack is a judge.
George says Jim is not a judge.
Gerry says Joe is not a journalist.
Jack says George and Jim have different jobs.
How many judges are there?

EXAMPLE 4. *To "Brighten up the holidays for them" Professor Proble gave his daughters an examination. To add to the "fun" he offered a prize for the best forecast of the result.*

These were the four forecasts:—
Rose: "Lily will be first, I will be second, Daffodil third, Violet fourth."
Violet: "I shall be top, Lily second, Rose third, Daffodil fourth."
Daffodil: "I shall be first, Violet second, Lily third, Rose fourth."
Lily: "Rose will just beat Violet for first place, I shall be third. Daffodil last."
Lily, with a place correctly forecast, won the Professor's prize. The other three girls were completely wrong.
In what order were they placed? (*Hubert Phillips* (1)).

This example calls for a new type of punching which can be used for many similar problems. We may call it $n!$ punching, and in this case we have $n = 4$. On 24 cards we punch the 4! permutations of 1234 in a code using 4^2 holes. The permutation 2314, for example, is coded 0100, 0010, 1000, 0001. (As before 0 denotes a slot and 1 denotes a hole). For this problem this card can be used to represent the possible solution, "Rose is second, Violet is third, Daffodil first and Lily fourth". It should be clear how to use the information to remove cards until finally just one is left, which gives the solution.

Most of the popular puzzles are of the 2^n or the $n!$ types; but it is always possible to combine the ideas and make up puzzles of the type $2^m n!$

Many puzzles of this kind involve derangements rather than permutations, that is they incorporate the well-known type of puzzle condition " . . . but no man lives at the town that bears his name." For this kind of puzzle we are clearly only concerned with the subset of cards associated with the derangements of n things. It is well-known that the number of derangements of n things, subfactorial n, is the nearest integer to $n!/e$; so if subsidiary tricks of coding are to be employed to build a luxury model card computer for this type of job it would be useful to have one extra hole coded 'yes-no' according to whether the permutation carried is a derangement or not.

Remembering the large numbers of holes available on commercial cards it can be seen that it is quite feasible to use different sets of positions on a pack of cards for different types of punching. Thus 128 cards will accommodate both 2^7 and 5! punching, and most people will not want to go beyond this. Such a pack is more than big enough to verify all of the theorems in the calculus of propositions (without quantifiers) as it is developed by such writers as Whitehead and Russell and Hilbert and Ackermann; but it is nothing like big enough for the prize puzzle in a recent Sunday Times, or the classic example on divorced couples which was set in a Cambridge scholarship paper.

Cards not only find the solution to a puzzle—they find *all* the solutions, and this is very important. Also, if the puzzle has no solution the cards quickly establish the fact while trial and error methods continue groping. Students who are greatly interested in puzzles will often be cynical about mathematical techniques for solving them; but when the question of setting puzzles is raised, and of seeing that there is only the one solution and that the clues are not contradictory or redundant they readily see that "common-sense" alone is scarcely adequate, and realise the need for a systematic, disciplined study of logic.

In the past the teaching of geometry was often defended on the grounds that it provided "a training in logic". Today this claim can no longer be maintained for two reasons. In the first place the present century has seen determined attacks on Euclid's geometry as a logical system, many weaknesses have been brought to light and so-called 'elementary' geometry is now seen as a logical structure which is comparatively complicated. Secondly, it has been realised by teachers that the formal approach to geometry found in Euclid is only for those who are already mathematically sophisticated, and that the subject must be introduced in a much more intuitive fashion. This intuitive approach is, in theory, formalised later on; but this formalisation is at the best only attempted in a half-hearted fashion, and it would be no exaggeration to say that the great majority of teachers who teach the subject could not give an account that was at all accurate of the shape which this later formalisation should take.

There are yet other reasons why the teaching of geometry is in a bad way at the present time, not only in this country but abroad as well. There is the feeling that it is a difficult subject, and that boys can be got through the examination by concentrating on other parts of the syllabus; there is the feeling that engineers and applied scientists can get along without geometry provided that they know enough algebra and calculus; and worst of all there is the accumulation of the effect because each generation of teachers that is bred in an atmosphere where the proper study of geometry is neglected is less equipped than the one before to turn the tide.

The teaching of geometry needs revitalising in its own right, and that is another problem, but it is certain that in future there can be no plea that a half-hearted formalisation of geometry should be in the syllabus in order to teach boys logical reasoning.

In future logic must be taught directly and not second-hand.

The revolution has already taken place in American college textbooks, which more often than not begin with chapters on logic and set theory before moving on to the calculus which many of our own pupils master quite easily at school. The wind of change is also blowing in Europe and this attitude finds support in France in the writings of Lucienne Felix and in the practice of other teachers who meet together at the International Commission.

We can see an approach developing in which the early stages of mathematics at school will be indistinguishable from the early stages of English, for the aim of both will be clear expression. Gradually principles of classification will emerge, and while the English teacher discusses parts of speech the mathematician will build the algebra of classes around adjectives and the few basic conjunctions. There will be a rich field of examples in problems of sorting, classifying and description. The consideration of puzzles along the lines described in the first article of this series will introduce the notion of logical necessity, and the algebra of classes which the students will have been developing can then receive a new interpretation as an algebra of propositions. All the while the students will be reasoning freely on circle diagrams, and perhaps with punched cards.

Naturally the student will also learn the algebra of numbers in its proper place, but there can be little question that Boole algebra is the more basic mathematical discipline, and it can show quite clearly, at the stage when the pupil begins to specialise in mathematics, what is meant by an abstract system. Boole algebra has many interpretations and so the abstract algebra is recognisable as the basic structure involved in the many concrete realisations; whereas the traditional algebra of numbers is a highly specialised algebra, so specialised that it only has the one interpretation —in terms of number. When a mathematical system is so specialised that it has only one interpretation it is much more difficult to see it in the abstract, because it seems of necessity tied to the one concrete system which embodies it.

Throughout the development outlined above the teacher can know quite clearly the direction in which he is moving. He is moving towards the foundations of the subject, and there is no reason at all why the culmination of the school mathematics course could not be the axiomatic formulation of large sections of the subject in terms of the notions of set theory and logic. (In this connection see reference (2)). Also he is moving towards the mathematics of logical design and automatic control —eminently practical subjects at the present time.

At every stage in this programme visual aids have a part to play. We have seen one application of Boole algebra in the design of electrical machines. The scrupulous examination of details in a logical proof can easily seem mere pedantry to the student who is anxious for his mathematics to achieve practical results. But when it is a question of designing machinery the scrupulous examination of details takes on a new light, because a machine cannot know what you mean, it can only carry out literally the instructions you feed in.

It may be difficult to convince pupils that they should be precise when talking to their teacher; it is rather easier to demonstrate the need for precision and clarity when talking to a machine.

Finally, logic is a traditional study for students of philosophy and the humanities; by promoting this noble study which links the schools of Ancient Greece with the industrial machines of today we may help to close the gap between the two cultures in our society of which Dr. C. P. Snow has been writing. This task is more important than the teaching of any single subject.

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1. *Hubert Phillips, Heptameron, Eyre and Spottiswoode,* 1945.
2. *Lucienne Felix, The Modern Aspect of Mathematics, U.S.A.*
3. *T. J. Fletcher, The Solution of Inferential Problems by Boole Algebra. Maths. Gaz. No.* 317, 1952, *pp.* 183-188
4. *Martin Gardner, Logic Machines and Diagrams, U.S.A.*

(This series is now concluded).

APPARATUS REVIEW

Solid Geometry Models Demonstration Kit. *Mathematical Pie Ltd.,* £2 19s. 6d.

This set consists of eleven drilled plastic balls and 21 metal rods in 2 lengths. The balls are coloured and, by reference to the colour code enclosed, the following models can be made: cube, cuboid, regular and irregular tetrahedron, octahedron, square and rectangular-based pyramids, and square and rectangular-based wedges.

It is claimed that the teacher can assemble one of these models in a few minutes, but with the new set used by the reviewer considerable time was taken in assembling, for instance, a regular tetrahedron. The base was reasonably easy, although one of the rods bent while trying to insert it in one of the holes. However, when the 'third dimension' is being constructed (namely the ball at the apex which carries four rods to the balls at the corners of the base) considerable dexterity is needed to complete the model. The rods are also a tight fit in the holes, and screwing them in is not very easy owing to the slipperiness of the rods. The colour code would take a long time to memorise and consequently the instruction sheet is of paramount importance.

It seems that only one model is possible at one time using one kit, so a quick change during a lesson, say from a tetrahedron to a pyramid, would involve some hazard.

The rods, as stated, are a very tight fit at first but presumably with use a comfortable fit is obtained. Perhaps a slight taper on the rods for the first quarter inch would overcome this difficulty. The bright colours of the drilled balls make for attractive models and ease in description and discussion. The dark and light green

balls and the white, buff and pink ones, too, need stronger contrasting shades of colour than is supplied at present; we understand this is having the consideration of the makers.

The models are large (rods are 1 ft. and $\sqrt{2}$ ft. in length) and have many uses for teaching purposes. Diagonal and interior line properties are adequately shown by using elastic and link pins. With the kit are provided also four link springs which are used to join rods together to form articulated figures. By using the elastic bands and link pins, diagonal properties of plane figures can be shown in a dynamic way.

This adds greatly to the versatility of the kit and together with its portability and easy storage, it makes a useful adjunct to the mathematics room. Unfortunately the price seems rather high.

S.W.B.

Cellograph Graph Sheet

Philip & Tacey Ltd., London S.W.6. Price 8s. *per sheet. Wax marking pencil holder and six leads in assorted colours,* 8s. 4d. *(incl. Tax); refill leads* 3s. 11d. *for* 12.

This valuable teaching aid was brought to our attention by one of our readers, and at our request the suppliers sent on a sample for test in the classroom; we are more than grateful to both. The graph sheet measures 30 inches by 40 inches and is printed in half inch squares with bolder divisions of 2½ in. and 5 in. squares. These are printed in pale grey on a special white plastic surface which appears to be overlaid on strong paper, and the edges are bound in plastic. The marking pencil holder is of black plastic with a pen clip attached, and pressure on the top opens the claws at the writing end so that one of the coloured leads may be inserted. These leads are of wax and are available in red, brown, blue, green, yellow and black.

Drawing on the Cellograph sheet is easy; the crayon flows freely giving a fine yet bold line. The makers claim that the drawing may be removed by rubbing over with a dry duster, and while we found this to be true, we discovered that the removal was much easier using a slightly damp cloth. The sheet itself may be rolled up and easily stored but it really needs fastening to some firm surface when being used, and some teachers may prefer to fasten it permanently to some hardboard.

In use during lessons we found that this was far superior to drawing graphs on a blackboard. The sheet is of similar proportions to an ordinary sheet of graph paper (6 in. by 8 in.), the squares are clearly seen in the classroom and do not interfere with the graph lines, children use the marking pencil with confidence and there is no mess on one's hands. We have nothing but praise for this material, and if the phrase is not new, it can seldom have been truer than when we say that no Mathematics Room is complete without one.

The same firm produces a series of geography maps in Cellograph, and also some interesting number apparatus made in the same material. We have not had an opportunity to review these, but details are contained in the Cellograph leaflets published by them and obtainable from 69-79 Fulham High Street, S.W.6.

C.B.

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SECONDARY MODERN SCHOOL MATHEMATICS — V
MATHEMATICS IN RELATION TO THE OTHER SUBJECTS

CLAUDE BIRTWISTLE

To capture a pupil's interest is to use one of the most important aids of the educative process. This is particularly true with the Secondary Modern child, and one of the most important ways of capturing the interest of such children is to show them the relevance of the work they are doing to some pursuit in which they are interested. This might take the form of a hobby, some particular career which they are hoping to take up, or the work in some other school subject which has a particular interest for them. The mathematics teacher should look to these, therefore, to provide him or her with material for the mathematics lesson.

There is very little hope, of course, that one can just produce appropriate material at very short notice, and just as the teacher will build up his collection of mathematical models over a long period, so a carefully kept notebook listing the mathematics of other subjects, hobbies and occupations will amply repay the time taken in its compilation. Such a book is never complete and would probably be most useful in the form of a loose-leaf file classified under appropriate headings. The introduction of this material into a lesson can vary from a single example used to illustrate some mathematical principle to a project based upon the mathematics of some particular job of work. An example of the former type is the use of a formula from science, while the latter may be represented by a complete survey of the school premises.

As far as school subjects are concerned there is a great deal of mathematics to be found in the sciences, in geography and in all the practical subjects and the teacher should endeavour to cover the different types of calculations involved in these subjects using the appropriate methods. The latter point is emphasized as many teachers of these subjects often use inappropriate methods of teaching these calculations. Because of this it is desirable that the subject teachers should be approached with a view to getting their co-operation in standardising method throughout the school. This point also raises the consideration of whether an attempt should be made to teach examples drawn from a subject at the same time as this material is being studied in the lessons of that subject; there are arguments for and against this procedure and the reader is left to consider these for himself.

Examples are given below of some of the mathematics arising from other school subjects. It is worth noting, however, that there are times when parts of other subjects may be borrowed and used for the mathematics teacher's own purposes. An example which immediately springs to mind is in the teaching of elementary algebraic equations. Too often examples on these are of the 'Mary had a certain number of apples and received five apples more . . . ' type, which are somewhat artificial. However, if one chooses to teach the principle of moments at the same time, one finds an endless number of interesting examples based on a practical principle. A graduated rod is suspended at its mid-point (a Meccano beam or a metre-rod will do) and equal weights are suspended on each side, clearly at equal distances from the mid-point. Suppose now that one of the weights is doubled; where must we place it? A few simple experiments such as this bring out the essential principle that on

each side of the 'balance' the product of weight and distance from the centre must be the same. This now leads to examples of the type

$$2 \text{ lb.} \times 9 \text{ in.} = 3\text{lb.} \times \text{how many inches?}$$

or

$$2 \times 9 = 3 \times x$$

The idea can then be developed to find what weight is required at a given distance, and further problems may include more than two weights.

Physics and Chemistry, of course, provide a number of good examples of formulae. Boyle's law can provide simple exercises in algebraic equations and also provide material for the drawing of graphs. The same applies to many other formulae, e.g. conversion of Centigrade to Fahrenheit, but an attempt should be made to give a more imaginative treatment than the usual examples on the transposition of formulae and substitution in a formula. For example, a graphical treatment often gives an idea of how the different quantities in a formula are varying with respect to each other, as well as presenting a useful means of checking results obtained by substitution.

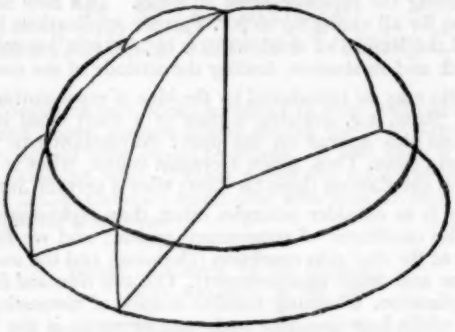
Science also provides some useful practice in the metric system. Many children are afraid of the system because of their lack of practice in it, and opportunities should be given for measurement of length, weight, capacity, etc. using these units. One important point is to ensure that the children have some idea of equivalents so that when they talk about a gramme they know they are talking about a very small quantity as compared with, say, a pound. The increase in the number of school parties going abroad should emphasize the importance of this work, and children will soon realize the need to know what to ask for when they wish to buy sweets or other things on the Continent, and also to know the cost of them. In fact, throughout the mathematics course, metric calculations should be introduced alongside English ones, as few children will go through life without having to use the metric system.

Mechanics can provide many useful mathematical applications. Moments have already been mentioned in simple cases, but there is a further use for these in trigonometry when the *perpendicular* distances of the lines of force from the fulcrum is not known; this necessitates the use of the sine or cosine to calculate the perpendicular distance. Sine and cosine are also used in resolving forces in two directions at right angles. Thus a force of 5 lb. wt. acting at 40° to some line OX may be resolved into two forces, 5 cos 40° and 5 sin 40°, in directions parallel to, and perpendicular to OX, respectively. These facts are easily illustrated experimentally. The opposite process, i.e. composition of forces, also requires the use of trigonometrical methods. Thus finding the resultant by calculation in the parallelogram of forces makes use of the cosine formula, while the same problem done by the triangle of forces makes use of the sine formula (Lami's Theorem).

Craftwork, too, can provide useful examples. Apart from the obvious examples on areas and mensuration of the circle, etc., one may find useful material for costing. Mathematical examples of so many pounds and ounces at so much a pound can often be artificial, but there are plenty of examples of this type to be found in industry, and in school it is always necessary to know the cost of a cake, a dinner, a dress, a wooden stool or a metal box which has been made. In industry the calculation is made simpler by the use of ready reckoners and adding machines, and there seems

to be no reason why these should not be used in classrooms. It is a useful exercise to make one's own ready reckoner; it can be a very simple form, or can be built up gradually into a comprehensive and useful booklet. The use of gears in workshops for the teaching of proportion is often overlooked. If there are no workshops, Meccano gearwheels can provide suitable examples. If two gears, A and B, are meshing, they revolve at speeds in proportion to the numbers of their teeth. As one revolves in one direction and the other in the opposite direction, strings attached to their shafts move inversely, but if a third wheel is introduced between A and B, the latter now rotate in the same direction.

Geography can provide many interesting examples of mathematics as applied to longitude and time; the principle is quite simple and the difference in time between two places whose longitude is given may easily be found. Examples may also be done on distances on the earth's surface involving latitude as well as longitude. A model is really required to illustrate the angles measured at the centre of the earth in order to obtain latitude or longitude, and a suitable one was described in *Mathematics Teaching*, No. 3. This issue is now out of print, and the diagram is reproduced again below. It consists of a circular wooden or cardboard disc on which is mounted vertically a semi-circular disc of the same diameter, together with two quarter circles. One of the latter (in front in the diagram) should be hinged at the centre and free to move over the horizontal plane. A wire circle, as used for lampshades, can be set in notches cut in the card to simulate a parallel of latitude.



There are also some useful film-strips published by Gaumont-British on *Latitude and Longitude.*

There are subjects not normally to be found in the school curriculum which can provide some useful mathematical material. To-day few of us can escape from statistics, and it is essential that children (and adults) should be able to understand the correct use of such material, and be able to detect its misuse. Appearing elsewhere in this issue is an article on elementary statistics which deserves careful consideration with a view to introducing this material into the mathematics course.

Surveying is another subject which seldom appears in the school curriculum, but provides a rich field of mathematical example. It is impossible in the space available to give details of a surveying course suitable for incorporation into the mathematics syllabus, but an indication will be given of the way in which trigonometry may be built up around practical surveying. After all, geometry and trigonometry grew up from the necessity to measure the land. How far one may go with the surveying and the trigonometry depends on the ability of the class, but all classes should be able to go some way along the course suggested.

The first approach to the subject could arise from the right-angled triangle and Pythagoras' Theorem; calculation of the diagonal of a rectangle, e.g. from corner to corner across the classroom. Further consideration of the right-angled triangle could develop the idea of similarity, and then the idea of the relationship between the sides of a right-angled triangle. An easy example to start with is the 45° triangle; the ratio of the sides may be calculated using Pythagoras. If a large 45° set-square is now made out of wood, it may be used for calculating heights of objects by sighting the top of the object along the hypotenuse of the triangle; thus the height of the object will be the same as the distance of the observer from it. One may repeat this with the 60° set square after first considering the ratio of the sides of this triangle. The limitations of these methods will soon be obvious and the desire for some means of measuring angles other than 45° and 60° may be made apparent. A simple theodolite may now be made by hinging a long piece of wood to the centre point of a blackboard protractor. The method of use is to hold the base of the protractor horizontal and sight along the movable piece of wood. This now necessitates the use of the tangent ratio for all angles up to 90°. Further applications here could include the calculation of the lengths of shadows cast by the sun, or conversely, knowing the length of a stick and its shadow, finding the altitude of the sun.

The cosine ratio may be introduced by the idea of representing a sloping surface on the horizontal plane, e.g. drawing a plan of a roof; what is the actual slope height?; how would this appear on the plan? Navigation can provide examples using both sine and cosine. Thus, given a certain course, what is the Northing (i.e. how far North) and the Easting (how far East) after a certain time.

The next step is to consider triangles other than right-angled. How shall we measure these? The conditions of congruency appear, and we find that surveying methods make use of the *three sides* condition (chaining) and the *one side and two angles* condition (base-line and angle measurement). The *two sides and including angle* condition has no application. Chaining consists simply of measuring the three sides by chain or tape, while base-line and angle measurement is the basis of the theodolite survey. Both methods may be practised. However, few shapes are simply triangular, and the need for methods of general surveying arises. This introduces the two ideas of *offsets* and *triangulation*. Offsets are distances measured from some straight line being surveyed to an actual line on the ground (e.g. a hedge). Perpendicular distances are taken at suitable points from the straight line to the hedge, so that points along the line of the hedge may be fixed on the plan, and the actual line of the hedge sketched in. The need for splitting a large area into triangles arises when one considers how one is to measure, for example, a rectangle in order to be able to draw it accurately; it becomes essential to make use of a diagonal, thereby dividing the rectangle into two triangles.

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Further topics may arise if one wishes to proceed along these lines. Thus the need to solve the general triangle can lead firstly to the sine formula and then to the cosine formula. Or again, the use of a prismatic compass in a traverse survey may introduce the closed traverse and the consideration of the angle sum of a polygon. Contouring is a more difficult and more technical aspect of the work. However, the above outline has been given to indicate the way in which it is possible to relate a great deal of mathematics to some practical subject with real life applications.

To revert to actual school subjects, it is possible to find mathematical interest in many unlikely places, and few school subjects are devoid of some such interest. History, for example, seems an unlikely source, but the historical approach to mathematics often appeals to many children. An article by the present author appeared in *Mathematics Teaching* No. 6 on the *Historical background* and gave examples of the use of historical material together with references to suitable books and display material.

The keen teacher can assemble a vast amount of material on the lines set out here, and will find that not only does it increase the interest of his pupils in the subject and show them the universal nature of the language of mathematics, but also gives the subject a sound cultural value.

THE LANGUAGE OF MATHEMATICS

DAVID S. FIELKER

"A spade is never so merely a spade as the word
Spade would imply."

C. Fry (Venus Observed)

The idea for this article arose out of a discussion with Mr. Birtwistle concerning a section of the original draft, of my article on Real "Real Life" Mathematics. Mr. Birtwistle, quite rightly, took exception to various calculations made on the assumption that "4 OUNCES" printed on an ink bottle carton could be Troy weight, Avoirdupois weight *or* Apothecaries' measure. The last, of course, was intended, and apparently any mention of ounces on a bottle means fluid ounces. The uninitiated, unfortunately, are not to know, especially when "114 ccs." appears elsewhere on the label, and with misguided glee I pounced upon my bottle of ink as an exercise in the calculation of densities! Mr. Birtwistle also pointed out that in a tannery "feet" of leather referred to square feet, and thus inspired I began to search for similar ambiguous and seemingly erratic uses of words in what is surely the most precise of the sciences.

As in the tannery example, ambiguity of dimension arises when the terminology is taken out of context, but presumably you are correct in these cases if you know what you are talking about! Thus builders mix a "yard" of sand, motor mechanics speak of an engine doing "5,000 revs.", allotment-holders dig up a "rod" of land and chemists make up your prescription in a "four-ounce" bottle. At the time of the

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talks on the Cyprus air bases one newspaper announced "Cyprus 'gap' is only 7 miles", which questionable terminology was shared by B.B.C. television. Such misuse has finally been recognised by Chambers's Dictionary, which shamelessly equates foot, square foot and cubic foot.

Sometimes what you are measuring is not so significant as where you are measuring it. In the north the gill is still a convenient measure for the modest drinker of beer; in the south it has long been relegated to the kitchen, and officially there are four gills to the pint. The mile in Britain and the States is 1760 yards, but travelling between the two it is 6082·66 ft; the German mile is given in Chambers's as "sometimes about $4\frac{2}{3}$ statute miles", the Irish mile 2240 yards and the obsolete Scots mile about 1976 yards.

Such technical groups as the woodwork fraternity avoid any ambiguity by inventing their own technical terms. A foot-run is a foot of wood, regardless of cross-section, a super-foot is a square foot, and a foot-super is a square foot of wood one inch thick. A square is 100 sq. ft. and a standard is 165 cu. ft.

A vocabulary like this has a very great advantage over most words that are used in elementary mathematics, for one of our difficulties in teaching is the confusion which arises in a child's mind when a word previously associated with some everyday object or item takes on a new meaning mathematically. In the teaching of fractions for example, even in the secondary school, we tend to take basic ideas for granted, forgetting that children will readily talk of the "bigger half" of an apple, and that for them the number of "quarters" in an orange depends on how many segments the orange has. Mathematical halves and quarters are abstract idealizations, and whether or not they exist concretely is a matter for debate at some appropriate time in every child's career. In any case approximation and exaggeration enrich our language, even if they play havoc with our mathematics. The phrases "101 things to do", "1,001 uses", "ten times better", even the word "decimation", all imply an exactitude that was never intended, and we "miss the target by a mile", "hang around for hours", get held up on the road home by "thousands of cars" and quench our thirst with "gallons of water" (though the stronger liquid adheres to its characteristic measure, the pint!). Even Chambers's Dictionary again is forced to acknowledge the inch as "a small distance" and the ounce as "a minute quantity". One can envisage serious conflict between the English and mathematics departments of any school as this goes on, and I am almost moved to rage by the football correspondent who writes that one team leads another on goal averages by "a decimal point", and the cricket commentator who mysteriously avers that a bowler is "a yard faster"!

I feel that as teachers of mathematics we must, with the dictionaries, accept figures of speech and the everyday use of mathematical terms, though we must point out definite errors where we come across them; but above all we must understand what goes on in the mind of a pupil when he or she is confronted with a familiar word in a new context. The field of geometry presents a few elucidating examples. Stupid though it may sound, it is hard for a dull London boy to realize that the words "area" and "volume" may mean something other than a piece of concrete in front of his house and a large book in the public library. It takes a lot of discussion, and a lot of geometry, to convince him that Leicester Square is not! Recently with a backward class of 14-year-old boys who had done no geometry before, I found myself going back to first principles, and it was harder to make them realize what a straight

line was than most 11-year-old classes. A horizontal or vertical line drawn on the blackboard was "straight", they agreed, but an oblique line was "crooked". Similar objections applied to a square made of meccano; when turned through any angle other than a right angle it automatically became a "diamond", however rigidly it was held. Yet how many of us take these things for granted?

As a secondary teacher I am afraid I become impatient with some of the terminology taught by primary teachers (who, I am sure, are not members of the A.T.A.M.!). The definition of an "oblong" applies equally well to a rectangle and to a sausage, and "corner" is far too vague a term to be given to an angle. It is in the teaching of fractions that one really comes across the folly of allowing the words "times" and "share" to replace "multiply" and "divide", and such answers as "You times it by five" must even annoy the English department!

As a final example, I have had long arguments with a colleague lasting on and off for three years over the word "sum". My argument was that a sum was the result of adding figures together, and so I objected to arithmetic in general being described as "sums", or any one problem or question as a "sum". He argued that subtraction was negative addition, multiplication was compound addition, thus division was negative compound addition, and hence all arithmetic was "sums". The dictionary let me down again, including among its many definitions "total: aggregate: result of addition: amount: quantity of money: a problem in addition, hence in arithmetic generally"! I compromised (since he was head of the mathematics department) by allowing my pupils to refer to anything as a "sum", provided that when challenged they could justify their use of the word, and also explain that "sum" was to addition as "difference", "product" and "quotient" were to the other three rules. All the same, when I hear a master addressing his bright class of twelve-year-olds with the words "Get out your sum-books" I somehow feel that the teaching in the school is not being carried out in the spirit of this modern technological age.

Am I being pedantic and fastidious? I think not, for precision is an essential feature of mathematics (even approximations are made to a precise degree of accuracy), and definition of one's terminology is an essential part of the progress of new mathematics. After all it was a mathematician who wrote

"*When* I *use a word*," *Humpty Dumpty said in a rather scornful tone*, "*it means just what I choose it to mean—neither more nor less.*"

"*The question is*," *said Alice*, "*whether you* can *make words mean different things.*"

Alice here is echoing the bewilderment of our pupils in suddenly finding mathematical rigour attached to the hitherto reasonably accurate and "one-valued" terms of their everyday existences. They are used to calling a spade a spade, and sometimes they call it a shovel; what they would never dream of doing is to apply the word spade to something entirely different. Yet that is what we ask them to do in practically every new piece of mathematics we teach. In which case some very clear explanation of this procedure ought to be necessary, and we must be very chary about what ideas and words we take for granted.

Building Up Mathematics

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B.A., Ph.D.
Lecturer in Mathematics in the University of Leicester

The teaching of Mathematics has been passing through an unhappy phase. It is not a popular subject: many children fail to cope at all adequately with it, and some are completely defeated. Dr Dienes is satisfied that the root cause lies in the earliest stages of the teaching of the subject. In this monograph he discusses his reasons for this view, and adds a great variety of practical suggestions for the cure of the problem. The book is provocative, deliberately so, and essential reading for every teacher (and potential teacher) of Mathematics.

Preface by Sir Herbert Read 16*s net*

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MOVEMENT AND PATTERN IN GEOMETRY

ALAN G. SILLITTO

Scottish teachers who have, in the post-Jeffery era, envied their English counterparts the freedom of manoeuvre they have been accorded, especially in Geometry, by the various manifestations of the "Alternative Syllabus," are on the point of having freedom thrust upon them. Next session, 1961-2, sees the first examination on the Ordinary Grade of the new Scottish Certificate of Education; it will be taken at the same age as the English "O Level," but, as secondary schooling in Scotland begins at 12+ and not at 11+, this means that it will be a Fourth Form, not a Fifth Form, examination. Some recasting of the syllabus has been necessary in all subjects, but the Scottish Education Department—in effect, the Inspectorate —who have the last word in these matters, have in respect of Geometry decided to go beyond the merely "necessary" in the direction of what they believe to be desirable.

What in fact we are to have is the Alternative Syllabus without the option! As from 1962, "Proofs of theorems will not be asked in the examination." "The Schools will thus have greater freedom to develop in their own way the background of knowledge and reference indicated in the syllabus . . . intuitive methods . . . symmetry . . . metrical treatments. Throughout the course pupils should be made familiar with logical argument and, in particular, they should understand that drawing and measurement do not constitute proof."

This is a big break with the formal Scottish tradition. To some of us it is exciting; to others, alarming; to all, a challenge. For help and guidance we look southwards, to the new crop of textbooks issued since Jeffery—and receive, in sum, a somewhat dusty answer. So many of the books indicate a weakening of the deductive links, without seeming to do anything towards the fabrication of a well-knit geometrical structure. B. I. Briggs in a review (M.T. Nov. 1957) lamented the tendency of textbooks in general to lead "in the geometrical field to a passive acceptance of relationships rather than to a real awareness." W. Flemming (Maths. Gazette, Oct. 1957) wrote: "The results of a recent G.C.E. examination suggest that the introduction of the alternative syllabus has reacted adversely on the teaching of Geometry. Many pupils give no sign of having acquired the feel of the subject."

We note the dangers; but how may we insure against them? Certainly, if we *only* weaken the deductive chain, the remaining structure will be the less viable. If we *only* subtract, the minuend must be diminished! It is still necessary—it is the more necessary—for the pupil to have the experiences which will help him to organise effectively his geometrical knowledge.

In Scotland, then, our task is *to make more impact* (to compensate for the decline in "rigour") *on weaker pupils* (the new Certificate is to cater for the top 30 per cent of the age group) *in less time* (four years instead of five).

The following notes tentatively outline one mode of approach, in which a special point is made of a wide-ranging "Stage A" study, high-lighting the part played by intuitive methods, making systematic use of symmetry and movement and exploring the geometrical content of large patterned configurations. In this

article, the main emphasis is on the intuitive, Stage A, work. In the course on which it is based, there is "reasoning" at every stage; to begin with it will involve the use of intuitively apprehended relationships (e.g. "construct a rhombus whose diagonals measure 8 cm. and 6 cm."), later there will be a more conscious examination of logical implication (e.g. an isosceles triangle has two equal altitudes, medians, angle-bisectors: which converses can my third year or my fourth year prove?).

Outline of a Possible Course

Euclid's Geometry begins with points and lines, definitions and axioms, but the Geometry of children begins with the material objects of the world around them. Euclid leads from parallelogram to the special cases of the square and rectangle; but it is these latter that children encounter first.

Commercial packaging provides cubes, cuboids, cylinders, cones, pyramids and triangular prisms; and an initial exploration of the properties of these solids may lead to their construction by the pupils, the nets, for simplicity, being frequently drawn on squared paper. Then attention may be focussed on the plane shapes from which the solids are made: first the *square* and *rectangle*, then the right-angled triangles which are their halves; two such "halves" give the isosceles triangle, and two isosceles triangles (with equal bases) give the *rhombus* if "identical" and the *kite* if not; halving the kite gives two identical scalene triangles, and these two "halves" can be recombined to give the *parallelogram*.

From this work may emerge—well in advance of any formal work on parallel lines or congruency—all the familiar properties of the special quadrilaterals, and the areas and angle-sums of these and of the special triangles and the general triangle. A deep appreciation of symmetry may be looked for, not simply in terms of folding, but in terms of the manipulative operations, of turning over and turning round, which "conserve" the figure.

If this is to be achieved, the pupils must do more than look at the blackboard, and draw lines in their jotters. They must have on their desks a supply of the figures under discussion. They must learn, with their hands as well as their eyes, that one parallelogram "template" can be placed exactly on top of another in just two ways (hence equality of opposite sides and angles, bisection of diagonals, equality of alternate angles—in a word, rotational symmetry); the kite in two ways; the rectangle and rhombus each in four ways, and the square in eight. Definitions of Euclidean type are utterly inappropriate; the properties of the figures are, not what the pupils can say about them, but what they can do with them.

At the same time the space-covering properties of the figures may be investigated. The square and rectangular templates may be used as tiles. So, less conventionally, may the rhombi and kites and parallelograms, and ultimately the scalene triangles. In all these tilings, parallel lines and equal corresponding and alternate angles impose themselves. If diagonals are drawn, rhombi appear among the rectangles, rectangles among the rhombi. Pythagoras is not far away.

These tiling figures are so rich in geometrical properties that it may be found profitable to return to some of them again and again during the first three or four years of the Geometry course; and if the early manipulative construction has been

given sufficient emphasis, a drawing of the tiling (on the desk or at the blackboard) will later suffice. By way of illustration we may look at the most general case: the scalene triangle tiling (though in class there is much to be said for approaching the general via the particular; for example, the properties of the special triangles—area, angle, sum, concurrencies, similarities including relative area—should in large measure precede those of the triangle in general). Fig. 1.

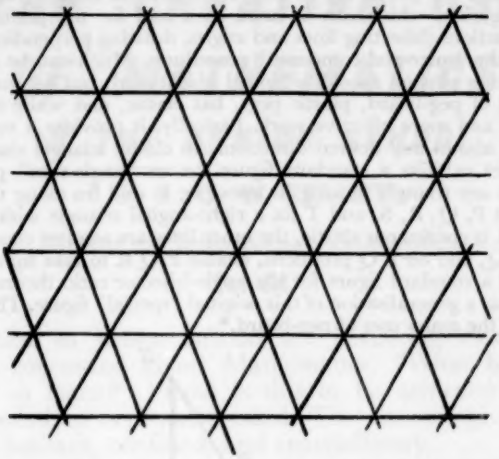

Fig. 1.

The list of properties which may be gleaned from this configuration is considerable. Equal corresponding and alternate angles, and supplementary co-interior angles, are all over the place. At every intersection we find, six times over, the three angles of a triangle summing to a straight angle. We see the equal intercepts made on any transversal by the equally spaced parallel lines; further, if along any transversal we pick out segments in say the ratio 2:3, we find the same ratio between corresponding intercepts on any other transversal. And as well as the elementary triangle—the original "tile"—we find all over the configuration equiangular triangles whose sides are 2, 3, 4 . . . times those of the original, and the areas of these (e.g., by counting "tiles") are 4, 9, 16 . . . times the original. (The "mid-point" theorem is a mere incident). If the concurrency of the perpendicular bisectors of the sides of a triangle has been accepted (for the "twice-magnified" triangle), the concurrency of the altitudes is asserted by the configuration, for the elementary triangle. And if we examine a "six-times magnified" triangle, and step from intersection to intersection along each median, from vertex to mid-point of opposite side, we may gain a new insight into the concurrency, and mutual trisection, of these medians.

At any stage of our elementary course, such a configuration provides a summary of previous knowledge, and a starting point for new exploration. "Mathematics," says W. W. Sawyer, "is the classification and study of all possible patterns."

The Geometry of tiles and templates outlined above is a Geometry of rigid shapes. (Note incidentally that angles occur only as fixed elements of shape; the angle as rotation will have its place, but it will also keep its place). But in due season we must consider the continuous deformation of a figure, for this also is an important weapon in the pupil's armoury. Templates and tilings need to be supplemented by deformable skeleton figures—a skeleton parallelogram with elastic diagonals, which exhibits en passant the properties of the rectangle; the skeleton kite or rhombus with elastic diagonals, which can be used as a tool for the performance of the standard constructions (bisecting lines and angles, drawing perpendiculars) before it is used to elicit the "ruler and compasses" procedures. Much can be done, too, with elastic and drawing pins on an old-fashioned blackboard; but for many purposes a good-sized sheet of peg-board, plastic pegs, hat elastic, and white string or wool, lead to speedier and more effective work. Basically, it provides a square grid, but yielding lines in almost any desired direction. An elastic triangle standing between two parallel lines exhibits a standard figure for an "angle-sum" proof; and *this* triangle becomes *any* triangle moving its apex-peg to and fro along the parallel. In Figure 2, pins at P, Q, R, S, and T fix a right-angled triangle with P T bisecting two sides. P Q R is continuous elastic, the other lines are wool or elastic. Other pins between P and Q, and on P Q produced, enable P Q R to take up a position such as P Q′ R; and a standard figure for the angle-bisector ratio theorem for triangle P Q′ R appears as a generalisation of this original "special" figure. These are merely two examples of the many uses of peg-board.*

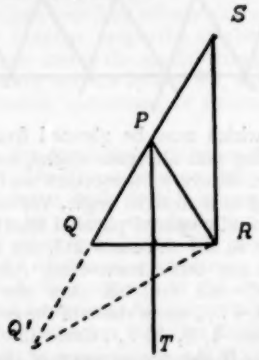

Fig. 2.

For obvious reasons, peg-board is not very helpful for circle work; more useful here is a board (wood, hard-board or plastic) with a circle painted on it and peg-holes at frequent intervals along the circumference and another at the centre. The

*Peg-board has obvious applications also to graphical work and Co-ordinate Geometry. It makes a real contribution in the field of Solid Geometry; using round wooden or plastic rods, string or elastic, and card, the relationships between lines and planes can be vividly exhibited; if a sheet of peg-board is supported horizontally and from three of the holes a length of elastic leads downwards to a small ring from which hangs a light weight, we have the basis of a good model for Desargues' Theorem in three dimensions.

AN IMPORTANT ANNOUNCEMENT

THE CREATIVE USE OF MATHEMATICS

IN THE JUNIOR SCHOOL

By **L. G. W. SEALEY**

Adviser for Junior Schools, Leicestershire

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Teachers in Junior Schools are becoming more and more concerned about Mathematics. "What has the child to learn?" "How is this to be achieved?" are the questions constantly asked. The answers given are often hesitant, confused, and contradictory.

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circular wooden discs used in basket-making have their uses; but sometimes one wants to use the surrounding plane, when tangents or secants are required. In fact, for most "circle" purposes one cannot do better than a chalk figure on a blackboard, with elastic chords, etc. where rotational movement is required. Two examples:—

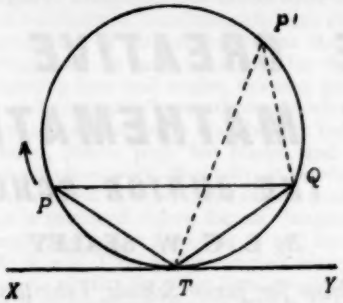


Fig. 3.

(i) Tangent-chord-angle theorem. (Fig. 3). In chalk, a circle, tangent X T Y at T, a chord P Q parallel to X T Y (preferably between X T Y and the centre), P T and Q T joined. Pins at T, P, Q; a loop of elastic stretched from T over P to Q. This symmetrical configuration reveals at once a special case of the theorem (so often the best approach!). Then the pin at P wanders to any other position P′ on the circumference; at once we see that angle QTY = angle QP′T; and almost at once that angle XTP′ = angle P′QT.

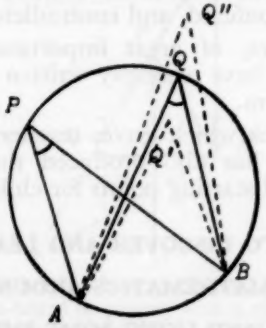


Fig. 4.

(ii) Converse of "angles in same segment equal." (Fig. 4).
In chalk: a circle; A, P, Q, B four points in order round circumference; chords AP, BP, AQ, BQ; angles at P and Q marked equal. Pins at A, Q, B; elastic stretched from A over Q to B. Pin Q moves to Q′ just inside the circle, angle AQ′B obviously exceeds angle Q or angle P; then Q moves to Q″ just outside the circle, angle AQ″B obviously less than angle Q or angle P. Summary: if Q is on the circumference, angle Q = angle P; if Q *not* on circumference,

angle Q $\neq$ angle P. So if Q = P, Q must be on the circumference. (Nearly all converses—Pythagoras is an exception—respond to this kind of treatment, which is particularly valuable in the classroom in providing support at the intuitive level for "reductio ad absurdum" proofs and in showing theorem and converse as an "equality" configuration between two inequalities, e.g.,

Q $\gtrless$ P).

Lastly, tracing paper. The pupil draws his figure in his notebook, and makes a tracing. He observes the effects of (a) translations: an angle, sliding along one arm, leads to parallels and equal corresponding angles; a triangle, sliding along a base-line, leads to an "angle-sum" configuration; (b) rotations: a certain rotation in the plane proclaims the equality of vertically opposite angles, the equality of alternate angles between parallels, and the essential symmetry of the parallelogram; and similarly we can explore the implications of the specific rotational symmetries of rhombus, rectangle, square, equilateral triangle, regular polygon, circle, etc.; (c) reflections: the tracing of isosceles triangle, kite, rhombus, rectangle, square, regular polygon, circle, etc. can be picked up, turned over and replaced on the original and the implications of this analysed—e.g., all the isosceles triangle concurrencies. For demonstration purposes, the teacher requires a sheet of cellulose acetate, and a chinagraph pencil. ("Change of axes" in Co-ordinate Geometry is another field of application of this simple device).

The emphasis in school Geometry has been changing. Beginners no longer need tread the Euclidean maze, and sometimes the loss to their mathematical training may be considerable; but if good use is made of movement, manipulation and patterned configuration, pupils may become more vividly aware of geometrical relationships in perhaps a more truly mathematical way, than has often been the case in the past. Was it not Whitehead who described Mathematics itself as "a series of aids to the imagination in the process of reasoning"?

Footnote: Some background reading on the Mathematics of Symmetry.

1. W. W. SAWYER, Prelude to Mathematics. (*Pelican*) *Pages* 97-102.
2. BIRKHOFF AND MACLANE, Survey of Modern Algebra. (*Macmillan*). Paragraphs 1-8 of Chapter VI on "Group Theory"; not of course at school level, but it is highly instructive to perform the transformations described (translations, reflections, rotations) using cardboard squares, rectangles, etc. in which no, or one, or two diagonals are drawn.
3. H. WEYL, Symmetry. (*Princeton University Press*). Discusses, and profusely illustrates, symmetry, in art and nature; and gives the underlying Mathematics unobtrusively. A small masterpiece.
4. VAN HIELE AND VAN HIELE-GELDOF, A method of initiation into Geometry at secondary schools. (*Article in a pamphlet "Report on Methods of Initiation into Geometry," edited by Freudenthal, and published in* 1958—*in English—by J. B. Wolters, Groningen*).
5. R. BRISAC, Exposé Elémentaire des Principes de la Géométrie Euclidienne. (*Gauthier-Villars*). For those interested in an abstract, axiomatic development of Geometry in terms of "displacements."
6. L. FELIX, Mathématiques Moderne∩Enseignement Elémentaire (*Blanchard*). A brilliant little book. Chapter VIII bears on "L'Espace," page 130 is perceptive and suggestive on Definitions.

MEASUREMENT IN THE PRIMARY SCHOOL

D. T. MOORE

Small children are interested in one thing at a time and perceive the object as a whole without reference to length or weight or volume. The transition from big or little, tall or short, etc. to formal ideas of measurement is a big and, philosophically, rather difficult step for a child to take. The need arises as soon as interest is aroused in comparisons and is developed by actual experience. In the early stages conventional units should not be used but the children allowed and encouraged to use whatever comes to hand. The use of a convenient rod, the span, foot or pace for measuring lengths is usual practice; it is much more difficult to devise a suitable individual standard for weight unless we are content with a simple 'lighter than' result although a handful of pebbles will often be available, and a measure of capacity is always available since milk has become a part of school routine. The need for a standardised unit eventually becomes clear and children should then be introduced to a standard unit of length, weight and capacity without references to the tables until much practical experience in the use of conventional measuring instruments has been obtained. The classroom should be provided with an ample supply of simple measuring instruments, yard sticks, foot rules (preferably not marked in inches at first), tape measures, pint and gill measures, scales and weights. Always emphasis should be laid on the idea of comparing with a standard, and experience of measuring weight and capacity should go hand in hand with measuring length.

Measuring area.

First ideas of measure will probably have arisen from arranging things in order of size; a series of rods placed in ascending or descending order by judgment of length, or a set of bricks of the same shape arranged in similar order (an intuitive idea of volume), but the idea of arranging things by reference to their surface area is unlikely to occur to a child and the concept has to be introduced by the teacher. Opportunity may be found in handwork lessons, e.g. making a lid to fit a box, or a door to fit the 'frame' in a model house. And there are on sale various games and puzzles which involve covering a space with a set of given shapes where the essence of the problem is to find the correct space coverage without regard to the thickness of the covering material. This sort of situation should be explored to the full. The space covered by two different shapes can be compared perhaps by cutting one up and fitting it on the other. Although it will be found simpler to deal only with rectilinear shapes it is important not to restrict the work entirely to regular figures. Next we can provide large numbers of squares, rectangles and triangles of identical size and use these to compare areas by coverage. The word area can be introduced at about this stage. Experiment followed by discussion will lead to the conclusion that a standard unit is necessary and that a square is the best shape, we can then introduce the square inch and experiment covering areas with these as a direct measurement. The area of a rectangle = length $\times$ breadth should not be taught, although if discovered it may be used as a short cut. Formal calculation in conventional units of areas of rectangular spaces is out of place in the Junior School.

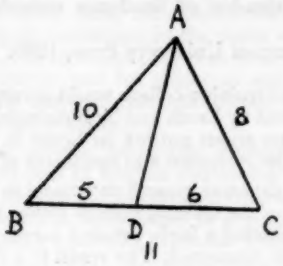
CORRESPONDENCE

Sir,

In his review of Larmour's trigonometry book in the October number Mr. Moore refers to the introduction of cot, cos and cosec as the tangent, sine and secant of the co-angle. The introduction of the six ratios at once is confusing to most boys. On the other hand there is much to be said for giving the historical derivation of the names in a passing reference if only to show that the trigonometrical and geometrical meanings of the words tangent and secant are closely connected, and as an illustration of the unity of mathematics.

Incidentally, secant was printed in Mr. Moore's review in error instead of cosecant.

I wonder how many people use a numbering notation like this, outside numbers denoting lengths running through and inside ones denoting those stopped at intersections.



i.e. BD = 5
DC = 6
BC = 11

Not foolproof, but useful at times.

Yours truly, E. J. G.

Sir,

In view of our Educational Policy we were more than interested in the winning entry in your "Planning a Mathematics Room" Competition. Our consultant, Dr. Kerr, Head of Department of Mathematics, Royal Technical College, Salford has on many occasions pointed out that a Mathematics Department needs more than blackboard and chalk, pencil and paper. No one would consider a Physics laboratory to be equipped without balances, etc. Why should we be content to have a maths. lab. without calculating machines?

Our policy has been designed to materially assist all Educational Establishments to obtain this equipment—by purchase, hire, experimental loan, etc. This is supported throughout the country by first-class mechanical service and free lectures, demonstrations, tuition, and instructional and application manuals.

May we, therefore, use the pages of your magazine to extend an open invitation to all your readers to write us on the subject. Our quarter century of experience of the requirements of modern industry and commerce are at their service to assist in the preparation of students of all ages for their part in building the Britain of the future.

Yours, etc.,

G. NORMAN PORTER,
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BOOK REVIEWS

Recruits to Teaching — A Study of the Attainments, Qualifications and Attitudes of Students entering Training Colleges

F. W. Land, Liverpool University Press, 1960. 83 pp. 7s. 6d.

Five or six years ago the training college world surveyed the attainment of their entries. Results were published by Leeds and by Birmingham Institutes of Education. The training college then set about putting its house in order and there has been a considerable advance in the reception and treatment of incoming students.

Dr. Land's book covers the same ground and comes to the same conclusions about the arithmetical equipment of the training college intake but his work is particularly valuable in that he has conducted a fairly detailed survey of the school backgrounds and attitudes of the students concerned. The result is a damning indictment of the mathematical education of the non-specialist for which both the grammar school and the university must share the responsibility. In fact, it would appear from Dr. Land's conclusions and data that arts graduates from the universities are as bad at mathematics and as antipathetic to it as are their counterparts on entering the training college; worse in fact, for most are 'reconditioned' at college but nothing transpires for the arts undergraduate.

Grammar schools still apparently advise girls to drop maths. at 13. Is this practice on the wane? 25% of the women did no maths. at school in their years before entry. Only 55% of the men and 73% of the women spent 5 or 6 periods per week in preparation for G.C.E. His conclusions are that the good results in G.C.E. appear to depend on at least 5 periods per week.

His results do not reflect the wide variation of qualification on entry between college and college, for one college at least has an average of between one and two A-levels for incoming students.

Students' attitudes are anything but pro-maths. Women rank it 10th and men 3rd. In social importance women rate it 10th and men 7th. In vocational importance both rank it 2nd. For good citizenship the students place it 5th. It is interesting that

geographers and historians rate their subject higher in vocational importance than do others but mathematicians take the general view. This finding is hardly surprising since maths. is 2nd anyway and it would be difficult for the mathematicians to beat the rest in their estimate of its importance; Dr. Land would find this significant.

The analysis of ability is a damning indictment of primary and grammar schools, the most damning in its cold figures of anything I have seen. Dr. Land draws the conclusion that to break the circle, the colleges must institute remedial classes. This is only a long term policy. It must take at least 20 years for the new intakes into the college to improve if one relied solely on this. The challenge is to the schools *now*. The grammar schools must discourage entry to colleges of anyone who is not mathematically competent to the very modest standard which is required of them. Perhaps if, next year, the colleges refused entry to all who failed to qualify mathematically, there would be such a shortage that the country would rise up in arms. One might comfort oneself that the analysis of errors of a sample of 100 students was not typical. Altogether in the survey were 1880 women and 374 men. But I am afraid published evidence only too sadly confirms Dr. Land. Students, asked to say what they thought they could do, expressed great confidence in their ability to do straight forward number work (alas, their performance did not bear this out), but were completely lacking in confidence in the main with anything beyond 2nd form grammar school work. It is not surprising that there is a significant correlation between the student's confidence and his choice of age group when it comes to teaching. This means that too many frightened, twisted, mathematical failures will prepare the children for the grammar school.

Dr. Land's conclusions appear reasonable enough. All intending teachers should have a G.C.E. O-level pass in English and maths. The Universities, battling with the bulge will skim off more of the intellectual cream. It would seem to me that if the best intellectual education is to be found in a grammar school, then to meet university expansion, we require a parallel grammar school expansion.

This report requires critical study, its statistics are a little involved and in at least one instance appear to be inconsistent. Nevertheless its main conclusion is inescapable: pre-college mathematical education is largely unsuccessful. Do head teachers conceal their students' deficiencies? Are the right students recommended for the job? Here is a grave national problem verging on scandal. Associations such as ours must try to do their utmost to mitigate it.

P.S. Would someone like to write a report on what training college students CAN do!

C.H.

Engineering Mathematics

J. Blakey & M. Hutton. Blackie, 1960. pp. 603. 40s.

This text-book covers the mathematics for a degree in engineering. It is substantially the same book as the well known text, University Mathematics by Blakey*. The chapters on polar equations of a conic and the sections on coordinate solid geometry have been omitted and replaced by chapters on The Numerical Solution of Equations, Statistics, Relaxation methods, and Operational calculus.

Numerical solution is concerned mainly with Horner's and Newton's methods and contains an application of Newton's formula to interpolation. There is sufficient for one to become proficient in the art of solving examination questions but it will not serve as an introduction to other books on numerical analysis. Statistics is covered in 27 pages dealing with mean and standard deviation, binomial and Poisson distributions. Relaxation methods are explained with reference to the solution of simultaneous equations and indications are given in the application of the technique to differential equations. Operational calculus applies the Laplace transform to differential equations, periodic functions, impulse functions and partial differential equations.

The authors are to be congratulated in once again providing a book which will be valuable to students concerned with obtaining a degree with the utmost administrative efficiency. It is not a book for the 'lone wolf' but, coupled with efficient interpretation, many a spare-time student will have cause to bless the writers.

C.H.

* *Reviewed in No. 10 (July 1959) page 39.*

Time for Sums

Elizabeth Bailey, Oliver & Boyd. Book I, 2s. 6d.; sum cards 1s. 9d. Book II, 2s. 6d.; sum cards 1s. 9d. Book III, 3s.; sum cards 4s. 9d. Book IV—in preparation.

Number combinations from 10 to 20 are built up, addition and subtraction facts being tackled side by side, in the third book of the series (the only one available to the reviewer). There is a set of five supplementary cards for practice work with each page. After the addition of tens and units with carrying, multiplication and then division by two is explained. This is quite a useful series if used intelligently by the teacher as a means of giving plenty of graded examples.

A.L.O.

Modern Mathematics with Numbers in Colour

C. Gattegno. Cuisenaire Company. pp. 84

This pamphlet on the use of Cuisenaire material is described as "a Manual for Primary School Teachers" and, as such, is intended to give the teacher a complete course of learning rather than to be a guide to teaching. It is obvious at first sight that this is not the sort of book that one settles down to read in an armchair, but should be studied with the material close at hand.

It is encouraging to see that the algebra of the coloured rods is considered first and is followed next by the arithmetic and then by the geometry of the material. Each of these three sections is given an exhaustive treatment and the arithmetic and geometry sections are clearly explained and well illustrated with concrete examples. One wonders, however, if the first impression of the algebra may not act as a deterrent to many prospective readers. There is no denying its algebraic value but much of the phraseology could have been considerably simplified.

It is a pity that Dr. Gattegno has apparently blinded himself to the existence of other apparatus used in teaching mathematics to young children and so gives the impression that, in his belief, the Cuisenaire material is the answer to a teacher's prayer. In his preface, Dr. Gattegno states that he knows that the answer to all the problems of teaching arithmetic have been found. This surely is too wild a claim for any one man to make and the unduly egotistical tone of both Preface and Introduction is hardly likely to encourage many potential readers to study this pamphlet with an open mind.

I. L. Campbell.

Naive Set Theory

Paul Halmos. Van Nostrand Co. Inc. pp. vii + 104, 26s. 6d.

"The student's task in learning set theory is to steep himself in unfamiliar but essentially shallow generalities . . . , general set theory is pretty trivial stuff really, but, if you want to be a mathematician, you need some and here it is; read it, about it and forget it."

This is a paraphrase of the last remark of the preface to the book. Not everybody will agree with the view expressed. Axiomatic set theory is definitely the province of the pure mathematicians and this book gives almost a 'stage A' of the more rigorous treatments to be found in advanced university texts. It has enough symbolism and sequential development to read like mathematics and yet is so full of discussion and clear explanation as to meet the needs of a good sixth former who wishes to specialise in mathematics, or a student in training college who wishes to learn the first things about the queer symbolisms which are used in mathematics nowadays.

After a brief treatment of the axiom of specification, the rock on which Frege almost foundered when Russell's famous paradox ('Is the class of all classes which are not members of themselves a member of itself or not!') was propounded, he treats of un-ordered pairs, unions and intersections of sets, power of a set, ordered pairs (without giving any of the common applications such as in topology or even complex numbers). Relationships, functions, inverses and composites are then discussed to be followed by numbers, the Peano axioms, arithmetic, order, Zorn's Lemma, well-ordering, transfinite recursion and the arithmetic of ordinal and cardinal numbers. The contents are written in a very readable manner giving the barest outline with few exercises and those merely an extension of the text. There are no 'real-life examples'; one would have welcomed more about the paradoxes of specification, more about 'why set algebra', more of the powerful method of recursive definition in function theory and perhaps an introduction to the part played by some of the ideas mentioned in philosophy courses. In fact the author limits himself to explaining the symbols and terms of set theory the first stepping stone of a long chain leading to axiomatics and real analysis, a step which his readers having once taken will want to extend and follow up.

Definitely a book for the library where students read mathematics with what goes on in universities in mind.

C.H.

The Teaching of Mathematics in the Secondary Modern School

Olive I. Morgan. Geo. G. Harrap, 1959. pp. 144. price 10s. 6d.

This book is most unfortunate in its date of publication. Had it appeared ten or fifteen years ago its review would have been far more easy. As it is, in this day and age, one feels it is out of touch with what is actually happening in Secondary Modern Schools. The book is designed to help non-specialists who are called upon to teach mathematics.

The first section deals with the content of the curriculum. Six objectives are outlined: to think abstractly, to train to reason, as a convenient method of assessing progress, to fill a need (immediate or remote), to help form orderly habits of presentation and to bolster a child's self-esteem. Then follows a list of topics commonly associated with the primary school but which Miss Morgan lays down for the secondary phase. It is true that many primary schools ignore these ideas but their number is lessening each year.

The foundations of understanding follow but there are no suggestions for looking at old knowledge as stage A of new. This section could be recommended to a primary teacher. Next comes 'the tools of arithmetic': four rules, money, fractions, decimals. The tools are sometimes prefabricated and the precept of understanding is often ignored in the face of expedience. The tools seem to have no connection with mathematics as a developing abstract science.

The last year of school life, it is suggested, should be devoted to such topics as pocket money, expenses, saving money, family budgets, employment, home, leisure and citizenship. The book closes with five pages on the backward child.

This would be a very dull way for the secondary modern pupil of 1960 to spend four or five hours per week for four years. Any idea that mathematics might become a fascinating hobby, that the great ideas which permeate mathematics are of fundamental importance in all thinking, anything creatively exciting, is so patently omitted that one wonders whether mathematics is worthy of so much time spent on it. As a book on the teaching of arithmetic, not referred to a specific age range in the stream of school life, this book has something to be said for it, but as a basis for the mathematical education of the middle 50% of the secondary population, one would with-hold it from anyone who was required to teach mathematics. C.H.

Pure Mathematics

F. Gerrish. Vol. I — Calculus, 360 pp. 25s.; Vol. 2 — Algebra, Trigonometry, Co-ordinate Geometry, 396 pp. 35s. C.U.P.

These books are designed to cover Part I of the London B.Sc. General Degree and most of the Pure Mathematics required for the Diploma in Mathematics of the Mathematical Association. They would appear to be suitable also for mathematics specialists in the VI Form of a Grammar School, and also for students specialising in mathematics at Training College.

The volume on calculus gives a rigorous treatment of limits, the derivative and integration, and follows with chapters on differential equations, theorems of differential calculus (Rolle, Lagrange, Taylor and Maclaurin, etc.), integration as a summation, geometrical applications of calculus and functions of several variables. Volume 2 deals with algebra of polynomials, determinants, series, complex algebra and theory of equations, De Moivre's Theorem, co-ordinate geometry of the straight line, parabola, ellipse, hyperbola and the general conic, polar equations, 3-dimensional co-ordinate geometry and spherical trigonometry.

It is not easy to understand why the author has chosen to split his two volumes in this way, as one must inevitably contain something of the other. However, both volumes contain a table of contents and an index, but the latter could be made more comprehensive by putting topics under more than one heading. For example, Mr. Gerrish's treatment of plane trigonometry is rather sparse and is scattered through the two volumes; it is not easily found. Thus if one wishes to refer to inverse circular or hyperbolic functions one finds no entry in the indexes under *trigonometry*, *hyberbolic*, or *inverse*, while *circular* has but one reference in one volume. It is not until one thinks of looking under *function* that one finds the references, and even then one has to look at both indexes before one gets the full references. This matter is especially important since, as the author admits, the topics will not necessarily be studied in the order given. Perhaps in future editions he will give a complete index to the *whole* work in *each* of the two volumes.

Although there does not appear to be a great deal which is new in Mr. Gerrish's approach, his treatment is sound and thorough. The student working partly on his own will find the discussion in the text comprehensive and rigorous. A large number of representative worked examples is incorporated and there is a good selection of exercises, many of them with hints on methods of solution.

This book is certainly worthy of examination and consideration by all taking students at these levels.

C.B.

The Fascination of Numbers

W. J. Reichmann, Methuen, 1958. pp. 178. Price 15s. net.

First published in 1958 and reprinted with minor corrections in 1958, this is a fascinating book for the school library or a school prize for a lower school boy who has begun to show an interest in mathematics. It is concerned with the properties of numbers (as its title implies) and covers such items as series (involving triangular, square and cube numbers, difference tables and the absorbing patterns presented by particular instances); primes, factors and divisibility; logarithms and trig. ratios; perfect numbers and oddities; recurring decimals; congruences; irrational, 'imaginary' numbers and continued fractions, closing with pseudo-telepathy, fallacies, magic squares and number mystery. The book is supplemented by appendices on Pascal's triangle the four fours, naming the day, series and weights problems and 'digital' roots of fractions.

There is an almost inexhaustible store of interesting problems with which to enrich algebra and arithmetic lessons for the discerning teacher. So often revision in the first forms is repetitive and sterile in its fruits yet, as Mr. Reichmann shows, the relations between numbers are exciting and demanding, so much so that the teacher who constructed his examples from the store of knowledge here would have a mathematically excited form who would be eager to learn more. Those who hope to begin a study of the theory of numbers will find a good stage A treatment although this is not the author's intention.

Buy it for your library, use its ideas and encourage the pupils to read mathematics by referring to it as private reading. All types of schools will find it a welcome addition to their stock of teaching aids.

C.H.

Paradoxes and Common Sense

A. J. Kempner, Van Nostrand Co. Inc. pp. 22. 3s. 6d.

This is a fascinating pamphlet which covers a wide variety of paradoxes ranging over many mathematical fields: logic, geometry, convergence, infinity, continuity and the parallel axiom. All the common fallacies and paradoxes are briefly but clearly treated, most of those given are very good talking points for classroom use.

The price is a little high when one compares the value offered by Pelican for many more pages, but the absence of verbiage, fourteen talking points well worth 3d. per point are good reasons why one could invest in this little booklet.

C.H.

An Introduction to Mechanics

A. E. Short. University of London Press Ltd. pp. 330 12s. 6d.

This is a textbook intended for students taking Mechanics as a subject at 'O' level of the G.C.E. As its name implies it is an introduction to the subject and a rigorous treatment is not attempted; the approach is definitely 'Stage A'. For example, no attempt is made to prove that resolution of forces in two directions and taking of moments about a point are necessary and sufficient conditions for equilibrium.

The work is clearly set out and the text is lucid and well-illustrated with good diagrams; there are plenty of worked examples and a good selection of problems to be solved, including 55 revision questions at the end of the book. The approach is a practical one and many experiments are described to illustrate the principles. This, in fact, makes the book an admirable introduction to the subject for the student who has not done any physics.

Many teachers may wonder at the author's arrangement of his chapters, but there seems no reason why they may not be taken in other sequences. Chapter XV on 'Varying acceleration and force' is the only one requiring the use of elementary calculus. The book concludes with chapters on 'Liquid pressure' and 'Gases'.

C.B.

La Matematica y Su Ensenanza Actual

P. Puig Adam, Publicaciones de la Direccion General de Ensenanza Media, Madrid. pp. 465.

This book is a collection of Puig Adam's writings over the years. Much of the material here is contained in earlier books of his which have been reviewed in this journal, but there are also a number of wider essays on mathematics and life which are not generally available.

The first part is on general principles. It commences with a chapter entitled "A Human View of Mathematics", continues with one on cybernetics and information theory, and concludes with essays on new developments in teaching method. The second part, mathematics teaching in action, brings together very many examples of practical lessons embodying that felicitous use of material from the world around us that was so characteristic of the author.

This was published before the author's recent untimely death, but it will now be a memorial to him; and in reading it his friends can recapture something of the spell of this universal man, uniquely gifted and a teacher without a peer.

T.J.F.

Individual Mathematics

Goddard and Grattidge, Schofield & Sims Ltd.

The publishers of this series of text-books which was reviewed in our last issue have written to us pointing out that a statement in that review may give the impression that the text-books were published without answers. In fact, the copies seen by our reviewer did not contain answers, but an edition is available with answers so that pupils may mark their own work. In addition, separate books of answers are available to meet the requirements of those teachers who prefer to allow one set of answers to a group of perhaps five or six pupils.

To correct any wrong impression that may have been given, we set out below the various editions and prices of these books:

Pupil's Books I – III (without answers)	each 7s. 9d.
Book IV (without answers)	8s. 6d.
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Book IV (with answers)	9s. 6d.
Teachers' Book	2s. 6d.
Separate books of answers to Books I – IV	each 2s. 6d.

Correction: Messrs. George G. Harrap, publishers of the book *College Algebra and Trigonometry* reviewed in our last issue, desire us to point out that the price of the book is 40s. and that there are 476 pages; not 17s. 6d. and 314 pages as stated by our reviewer.

A.T.A.M. DIARY

1961

March 4th — Manchester School of Education: Middle ability range in Secondary Schools. Details—Miss D. Clutten, 14, Bristol Street, Burnley.

March 11th — Middlesex Branch: Repeat Exhibition of Aids: Institute of Education, London. Details—from Institute.

March 11th — Cambridge Branch: Visual aids: Swaffham Bulbeck School. Details—N. Reed, Village College, Swavesey, Cambs.

March 25th — E. Suffolk Branch: Geometry from 5 to 18. Details—Mr. Hampton-Wright, School House, Theberton, Leiston, Suffolk.

March — Probable meeting Maidstone, Kent Group. Details—D. T. Moore, 234 Birling Road, The Groves, Snodland, Kent.

April 11th to 14th — A.G.M. See page 19.

May 6th — Harrogate. Secondary School Mathematics. Details—D. T. Moore (as above).

ANNOUNCEMENTS

Miss F. Gross has drawn our attention to the fact that the Association would be performing a useful service in compiling and publishing a list of posters and booklets published by commercial firms which prove useful as teaching aids in the classroom, and she has agreed to compile such a list for publication in a future issue. Would all members who know of such materials please send full details to Miss Gross at 41 Newton Road, Bayswater, London W.2. We would particularly ask those who write to list all materials they know, even though they themselves may think they are well-known.

INCOME TAX RELIEF

Members are reminded that subscriptions to the Association for Teaching Aids in Mathematics may be claimed as allowance against Income Tax. The details should be entered on the usual Income Tax Claim Form, but where this is a first claim, the appropriate details should be filled in on form P 358 (obtainable at local tax offices).

SUBSCRIPTIONS

Members are reminded that the subscriptions to the Association are now overdue, and are asked to send these to the Treasurer at once, if they have not already done so. Prompt renewal reduces administrative costs as well as assisting the work of the administrative officers. Membership costs 10s. per year, which includes copies of *Mathematics Teaching* as published.

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